PARTICLE PHYSICS
AND
COSMOLOGY

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Chapter l. Quarks, leptons and forces

The fundamental particles of matter are *quarks* and *leptons*. They can be arranged in three *families* or *generations* according to the following scheme:

$$
\begin{array}{cccccc}
 & & & \text{Electric charge} & \\
\begin{pmatrix} u \\ d \end{pmatrix} & \begin{pmatrix} c \\ s \end{pmatrix} & \begin{pmatrix} t \\ b \end{pmatrix} & \begin{matrix} +2/3 \\ -1/3 \end{matrix} & \text{Quarks} \\
\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} & \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} & \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} & \begin{matrix} 0 \\ -1 \end{matrix} & \text{Leptons}
\end{array}
$$

The electric charge is given in units of the fundamental unit charge e = $1{,}602 \cdot 10^{-19}$ C.

The quarks are denoted by letters being a short for their names u = up, d = down, c = charm, s = strange, t = top, b = bottom.'

The leptons in the bottom line are the *electron*, the *muon* and the *tauon*. The leptons in the previous line are the corresponding *neutrinos*. In the standard model that we present here they have mass zero.

For every particle in the scheme there is a corresponding antiparticle that is usually denoted by a bar over the letter, for instance anti-u is denoted $\overline{u}$ and has opposite electric charge. The antiparticle of the electron, e^+, is the *positron*, the *antimuon* and *antitauon* are μ+ and τ+ respectively. There are also corresponding antineutrinos.

All these particles have spin 1/2 and are *fermions*, which means that they follow the *Pauli principle: Two identical particles cannot be in the same state.*

The quarks can, via the *weak interaction*, be converted into each other *within* the family and, with smaller probability also between the families. Only the first family of quarks is thus stable. The leptons *cannot* be converted between the families in the model we present.

The quarks do not exist as free particles; they are always bound to each other, either in pairs of quark-anti-quark (*mesons*) or in triplets of quark-quark-quark (*baryons*), alternatively anti-quark-anti-quark-anti-quark (antibaryons). These quark combinations will always have integer charge.

Quark-antiquark pairs have integer spin (0, 1, 2...) and are *bosons* (they do *not* obey the Pauli principle) while the triplet combinations have half-integer spin (1/2, 3/2, 5/ 2...) and are *fermions*. Mesons and baryons have a common name; they are *hadrons* (they interact strongly).

The tables below show the quark content of some hadrons

Spin 0 mesons

Hadron	Quark content	Charge
π^+	$u\bar{d}$	+1
π^-	$d\bar{u}$	–1
π^0, η, η'	$u\bar{u}, d\bar{d}, s\bar{s}$	0
K^0	$d\bar{s}$	0
$\bar{K}^0$	$s\bar{d}$	0
K^+	$u\bar{s}$	+1
K^-	$s\bar{u}$	–1

Spin 1 mesons

Hadron	Quark content	Charge
ρ^+	$u\bar{d}$	+1
ρ^-	$d\bar{u}$	–1
ρ^0, ω, ϕ	$u\bar{u}, d\bar{d}, s\bar{s}$	0
K^{*0}	$d\bar{s}$	0
$\bar{K}^{*0}$	$s\bar{d}$	0
K^{*+}	$u\bar{s}$	+1
K^{*-}	$s\bar{u}$	–1

Spin 1/2 baryons

Hadron	Quark content	Charge
n	udd	0
p	uud	+1
Σ^+	$u\bar{u}, d\bar{d}, s\bar{s}$	+1
Σ^0, Λ^0	uds	0
Σ^-	dds	–1
Ξ^-	dss	–1
Ξ^0	uss	0

Spin 3/2 baryons

Hadron	Quark content	Charge
Δ^-	ddd	–1
Δ^0	udd	0
Δ^+	uud	+1
Δ^{++}	uuu	+2
Σ^{*-}	dds	–1
Σ^{*0}	uds	0
Σ^{*+}	uus	+1
Ξ^{*-}	dss	–1
Ξ^{*0}	uss	0
Ω^-	sss	–1

Most of these hadrons are unstable and decay into particles with smaller mass. A table at the end of this book shows the most common decay channels and the corresponding lifetimes. You will also find a corresponding table for the leptons.

In Nature there are, as far as we know now, *four fundamental forces*:

- *Gravitation*, which can normally be neglected in the micro cosmos.
- The *electromagnetic* force that acts between all *electrically* charged particles.
- The *weak* force that can convert between quarks and between leptons. The weak force is for instance responsible for β-decay in atomic nuclei. The weak force cannot change a lepton from one family to another (in this theory). On the other hand the weak force can, with a reduced probability, change quarks between the families.
- The *strong* force, which only is felt by the quarks and then ignores the type of quark. It cannot, as the weak force, convert a quark into a different quark. It is the strong force that keeps the pairs and triplets of quarks together and also keeps the nucleons together in an atomic nucleus.

In order that two particles will interact they have to know of the existence of each other. In all modern theories of interaction the force between two particles is due to the exchange of *force particles* that carries information between the interacting particles. In the *electromagnetic* interaction this particle is the *photon*, γ, (massless, with no charge and with spin 1). In the weak interaction we have *three* force particles, *vector bosons*, W^+, W^- and Z^0 (with masses of the order of 100 proton masses and spin 1). In the strong interaction we have 8 *gluons* (massless, uncharged and with spin 1). In gravitation we have the graviton (uncharged, massless and with spin 2). You will find a table of the force particles at the end of this book.

Repeat the sections on relativistic kinematics from earlier courses. Especially important are the definitions of (total) energy, momentum, invariants and applications of the theory on relativistic collisions and decays. Note that all masses that we use are *rest masses*. Repeat also earlier sections on quantum mechanics, especially the *uncertainty relation*.

Review:

- Write down the scheme for the quarks and leptons. Give the respective electric charges. Which particles can interact electromagnetically, weakly and strongly?
- Study the tables at the end of the book. Typical times in the weak decays are 10^{-6}–10^{-10} seconds, in an electromagnetic decay 10^{-16}—10^{-20} seconds and in a strong decay < 10^{-20} seconds. Classify the different decays in the table with the help of this.
- Check that the electric charges of the component quarks add up to the correct total charge of the hadrons in the tables.

Problems:

1. A π° meson decays at rest into two photons. This is an electromagnetic decay. Why? (Give several possible reasons). Compute the energies of the photons.
Answer: 0.068 GeV.

2. A *K*-meson decays at rest into π^+ and π^-. Compute the energy and momenta of the π mesons.
Answer: 0.249 GeV and 0.206 GeV/c.

3. In 1974, the Ψ meson, the first particle that contains the charm quark *c*, was discovered that Ψ consists of a $c\bar{c}$ pair and has mass 3.1 GeV/c. It was observed when it decays into $\mu^+\mu^-$. What is the momentum of the muons and how far do they travel (in average, they are unstable and decay within a short time) if the Ψ meson decays at rest.
Answer: p = 1.54 GeV/c; distance = 9.6 km.

4. Two particles with masses m_1 and m_2 and the same momentum *p* move between to detectors that are placed at distance *L* from each other.
a) Show that the difference in time of flight for the particles to move between the detectors is proportional to p^{-2} if the momentum $p >> mc$.
b) Compute the least value that *L* can have if you would like to differ between a π mesonand a K meson with momentum 3 GeV/ c and the time of flight can be measured with a precision of 200 ps. The *K*-meson has mass 494 MeV/c, the π meson has mass 140 MeV / c2. Answer: 5 m.

Chapter 2. Interactions

Particles can interact with each other and particles can decay. A particle interaction can be written symbolically by for example

$a+b \to c+d+e$ or $ab \to cde$

A decay can be written

$a \to cd$

Examples:

$$\pi^- p \to \pi^0 n$$

$$\pi^+ p \to K^+ \Sigma^+$$

$$\Sigma^0 \to \Lambda^0 \gamma$$

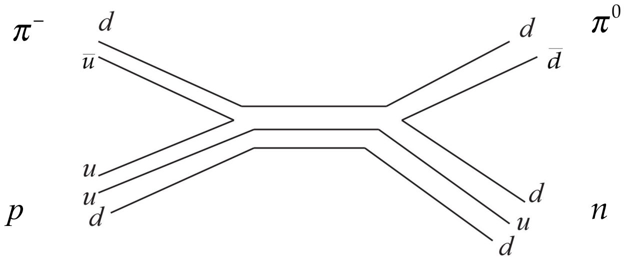
$$\Sigma^+ \to \pi^+ n$$

$$\mu^- \to e^- \overline{\nu}_e \nu_\mu$$

For all such reactions you have some rather simple conservation rules:

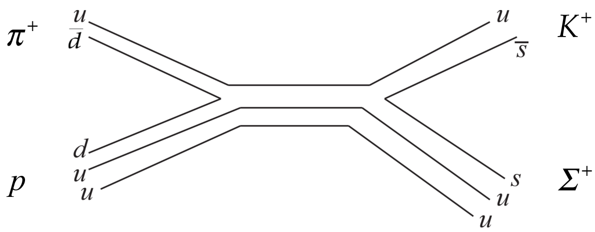
1. The electric charge is always conserved. Check that this is the case for the reactions above!
2. The quark number = (number of quarks - number of anti-quarks) is always conserved. This together with the fact that quarks can only group in quark—antiquark pairs and triplets (quarks or antiquarks) as given earlier, imply that baryon number is always conserved. Check all the reactions!
3. The lepton number (number of leptons - number of anti-leptons) is conserved by family in all reactions. This means that you can only create or annihilate a lepton by at the same time creating / destroying an anti-lepton. Check the last reaction!
4. Under strong and electromagnetic interaction the identity of a quark is not changed i.e. it cannot be changed into another kind of quark. However, this happens in the weak interaction. Look for example at the decay of Σ^+ where an *s*-quark is converted into a *u*-quark via the weak interaction. Check the quark content on both sides of the decay!
5. In a decay via the weak interaction, the quarks are normally converted vertically in the scheme, alternatively in zigzag between the families as for example $u \leftrightarrow s$, $u \leftrightarrow d$, the heavier quarks often decay in a cascade like $t \to b \to c \to s \to u$. This is because the force particles *W* have electric charge.
6. The reaction must be allowed energetically. Thus for example a particle cannot decay into particles with a total mass larger than the mass of the parent particle. The reaction $p \to ne^+\nu_e$ fulfils rules 1, 2, 3, 4, 5 (check!) but not 6 and this decay is not possible.
7. If a photon is involved, the interaction is electromagnetic or is partly electro-magnetic.
8. In some cases a reaction can occur with several kinds of interaction (and sometimes in several ways with one interaction). We then choose the dominant reaction in priority strong, electromagnetic, and weak. The weak interaction thus is only chosen if strong and electromagnetic interaction is forbidden.

We can write a quark diagram in order to simplify the analysis.
The first reaction above will look like



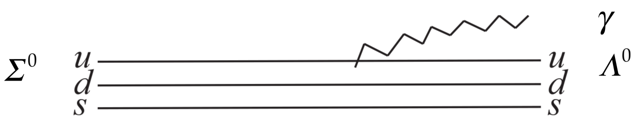
You see that effectively a u-quark pair annihilates and a d-quark pair is created. This is a rather typical strong interaction reaction.

The second reaction will be



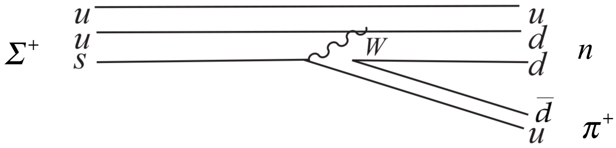
In this reaction you annihilate a d-pair and create an s-pair and it is obviously a strong reaction. Both the particles in the final states have strangeness, the sigma has strangeness equal to -1, the *kaon* (K-meson) has strangeness +1. This implies that via the strong interaction you have to produce pairs of opposite strangeness or that strangeness is conserved. This is also a strong reaction.

The third reaction is



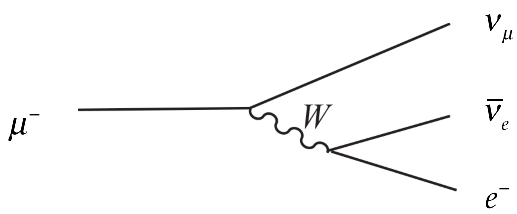
and is obviously electromagnetic. This reaction would be difficult to see in for instance a bubble chamber, as all the particles are uncharged!

The fourth reaction will be



Here we convert an s-quark into a u-quark via weak interaction, besides we create a pair of d-quarks.

The last reaction can be depicted as follows:



Note that we cannot destroy the "μ family" that survives as a μ neutrino. Also when creating an electron, a complementary member of the "electron family" has to be created.

Review:
Check that you understand and remember the conservation rules.

Problems:

1. Show by trying to draw quark diagrams for the reactions

$$\pi^- p \to K^- \Sigma^+$$

$$\pi^- p \to \pi^- \Sigma^+$$

that these reactions are very improbable. Why?

2. Classify the following reactions in strong, electromagnetic and/or weak or not allowed. Then choose the dominant one. In some cases there is more than one solution possible.

a) $\pi^- p \to \pi^- \pi^+ n$

b) $\gamma p \to \pi^+ n$

c) $\nu_\mu n \to \mu^- p$

d) $\pi^0 \to e^+ e^- e^+ e^-$

e) $\nu_\mu p \to \mu^+ n$

f) $p\bar{p} \to \pi^- \pi^+ \pi^0$

g) $K^+ n \to K^0 p$

h) $\tau^- \to \pi^- \nu_\tau$

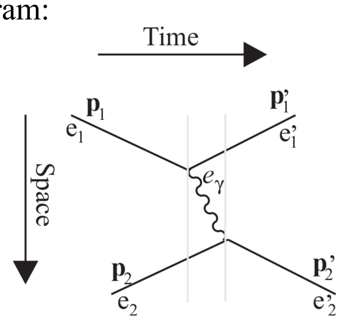
i) $D^- \to K^+ \pi^- \pi^- \quad D^- = (d\bar{c})$

k) $\Lambda^0 p \to K^+ pp$

3. Draw diagrams for the allowed reactions.

Chapter 3. Quantum electrodynamics (QED)

We can represent a reaction process where for instance two electrons interact by a space-time diagram:



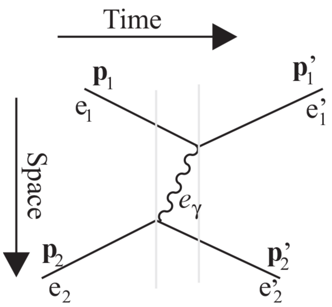
The upper electron emits a photon that after a while is absorbed by the lower electron a certain time interval (between the grey lines) we have an *intermediate* state that strictly is not allowed by energy conservation but due to the uncertainty relation, can exist for a short time. We can express the quantum mechanical *amplitude* of this process by

$$X \propto (-e)\frac{1}{E_n - E_i}(-e)$$

where $E_i = e_1 + e_2$ and $E_n = e_1' + e_2 + e_\gamma$ i.e. $E_n - E_i = e_1' - e_1 + e_\gamma$

The factors (–*e,* the electron charge) come from the upper and lower vertex respectively and give the *coupling strength*, a measure of the probability that a photon is emitted and absorbed respectively. The factor in the middle gives the time that by the uncertainty relation allows the photon to exist. E_i and E_n are the energies of the initial and intermediate states respectively.

The scattering process can *also* be described by the diagram



with amplitude

$$Y \propto (-e)\frac{1}{E_n{'} - E_i} + (-e)$$

where now $E_i = e_1 + e_2$ and $E_n = e_1 + e_2' + e_\gamma$ i.e. $E_n - E_i = e_2' - e_2 + e_\gamma = e_1 - e_1' + e_\gamma$, the last equality using the energy conservation $e_1 + e_2 = e_1' + e_2'$.

According to the rules of quantum mechanics we have, in order to get the reaction probability (or the cross-section), first to add these amplitudes and then take the modulus square of the sum. This gives

$$X+Y \propto (-e)\left[\frac{1}{e_1'-e_1+e_\gamma}+\frac{1}{e_1-e_1'+e_\gamma}\right](-e)=(-e)\left[\frac{2e_\gamma}{e_\gamma^2-(e_1'-e_1)^2}\right](-e)$$

The factor $2e_\gamma$ will disappear if we do a correct field theoretical calculation. We now put

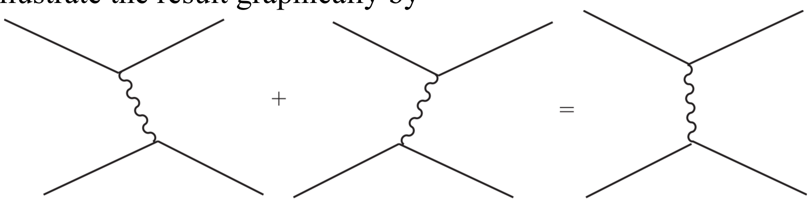
$$e_\gamma^2=\mathbf{p}_\lambda^2c^2+m_\gamma^2c^4$$

and get

$$A=X+Y \propto (-e)\left[\frac{1}{\mathbf{p}_\lambda^2c^2+m_\gamma^2c^4-(e_1'-e_1)^2}\right](-e) \propto (-e)\left[\frac{1}{m_\gamma^2c^2-P^2}\right](-e)$$

where *P* is the four-momentum of the exchange photon i.e. $P=P_1-P_1'$ with $P_1=\left(\frac{e_1}{c},\mathbf{p}_1\right)$ and $P_1'=\left(\frac{e_1'}{c},\mathbf{p}_1'\right)$. Note that the denominator is a relativistically invariant expression. We have here given the exchange particle, the photon, a mass m_γ in order to get a general result that can later be used also in the case when the exchanged particle is not massless.

We illustrate the result graphically by



The last diagram where the exchanged particle goes vertically is thus the sum of the two time-ordered diagrams to the left. It turns out that we *always* get such pairs of amplitudes and that they always sum up to something “simple”. This means that we can always directly use the diagram to the right and via simple rules can translate the diagram into a mathematical expression. Below we will give these rules (and they will always work):

• Every vertex in a diagram corresponds to a factor (–e) in the amplitude.
• Every internal line (that describes a particle exchange) corresponds to a factor

$$\frac{1}{M^2c^2-P^2}$$

where *P* is the four-momentum and *M* the mass of the particle that corresponds to the internal line.

It was *Richard Feynman* who introduced this graphical way of representing an interaction. Such diagrams are therefore often called *Feynman diagrams*.

Note. In a complete theory also the external lines will contribute with certain factors (for instance describing the spin) that we neglect here. It is precisely some of these factors that cancel the factor $2e_\gamma$. In the calculations that we will make, those factors will not be essential.

We can now estimate the e^-e^- cross section by taking the modulus square of the summed amplitude above. As the result is relativistically invariant we can evaluate it in the centre of mass (COM) system where $e_1' = e_1$ and $|\mathbf{p}_1' - \mathbf{p}_1| = 2p\sin\frac{\theta}{2}$, θ being the scattering angle. Inserting this and putting $m_\gamma = 0$ we get

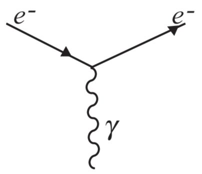
$$\frac{d\sigma}{d\Omega} \propto |A|^2 \propto \frac{e^4}{p^4 \sin^4 \frac{\theta}{2}}$$

As you may remember this is precisely the classical expression for the cross-section for scattering in a Coulumb field. Our rules give us the correct result and give us an interaction force that is proportional to $1/r^2$.

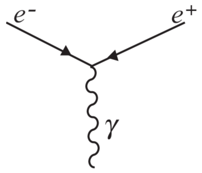
In the quantum electrodynamics (QED), diagrams with more that one internal line (more than two vertices) will correspond to higher powers of the dimensionless factor $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx 1/137$ in the amplitude. This factor is a small number that means that such diagrams can be neglected in a first approximation.

More about vertices

In the reaction that we studied above a vertex looks like:

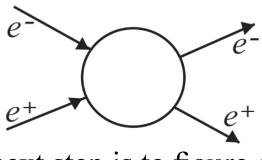


The nice thing about the diagram technique is that we can now reverse the arrows and thereby switch the particle that is represented by the line with the corresponding antiparticle. Thus we can transform our vertex to

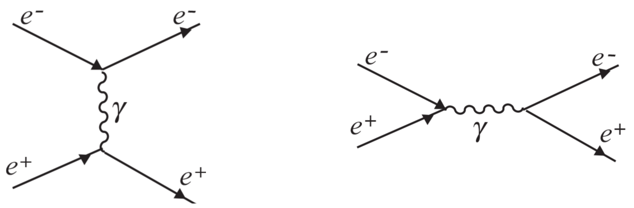


and so write down amplitudes for a great number of other reactions. Note that a vertex always consists of one photon line and two electron / positron lines. The electric charge is conserved in a vertex.

As an example we can study the scattering of an electron against a positron. We first draw this as a "bubble" diagram".



The next step is to figure out what is inside the bubble. We can use vertices and internal lines, each vertex must conserve charge and also we usually only want to consider first order diagrams. Some thinking gives the following allowed diagrams:



Note that each of these diagrams is the sum of two fundamental time-ordered diagrams. We indicate this by having the internal photon line either vertical or horizontal.

If we now use our rule for translating a diagram to mathematical expression, the left diagram will correspond to the amplitude

$$A_1 = (-e)\frac{1}{-P^2}(-e)$$

where P is the *difference* between the momenta if the incoming and outgoing electron.

The diagram to the right gives

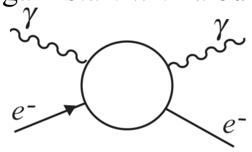
$$A_2 = (-e)\frac{1}{-Q^2}(-e)$$

where Q is the *sum* of the momenta if the incoming electron and positron.

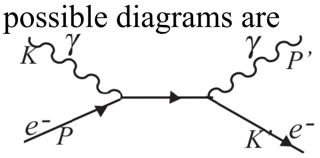
The total amplitude is the sum of these two amplitudes and the cross-section is proportional to the square of the modulus of this sum.

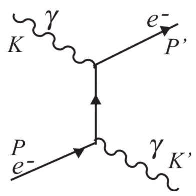
Compton scattering: Scattering of photons against electrons

We again start with a bubble diagram



The possible diagrams are





with amplitudes

$$A_1 = (-e)\frac{1}{m_e^2c^2-(K+P)^2}(-e) \qquad A_2 = (-e)\frac{1}{m_e^2c^2-(K'-P)^2}(-e)$$

respectively.

In both these last examples it turns out that our calculation is not very meaningful as it will be important to take into account kinematical factors from the external lines that we have neglected. However, we can estimate the Compton cross section by dimensional analysis. The cross-section has dimension length squared. The only factors that we have at our disposal is $\hbar$, c, ε_0, and m_e. We also know that we will have the cross section proportional to e^4 as we have two vertices. If we introduce the dimensionless parameter $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c}$, mentioned before, it turns out that the only combination giving the correct dimension to the cross-section is $\dfrac{d\sigma}{d\Omega} \propto \alpha^2 \left(\dfrac{\hbar}{m_e c}\right)^2$.

The quantity in the bracket is the Compton wavelength. The total cross-section will be the differential cross-section above, integrated over all angles. A more accurate calculation that includes the spin of the particles will give

$$\sigma = \frac{8\pi}{3}\alpha^2 \left(\frac{\hbar}{m_e c}\right)^2 \approx 6.5 \cdot 10^{-29}\,\mathrm{m}^2$$

Using the definition of the cross-section means that if we have a single electron and a flux of one photon per square meter and second, the probability of scattering is $6.5 \cdot 10^{-29}$. We get the same probability if we have one electron per square meter and a single incoming photon. If we have *n* electrons per cubic meter in a layer with thickness *l*, the scattering probability will increase by a factor $n \cdot l$ and will be $nl\sigma$.

The probability will be 1 for a thickness given by $l = \dfrac{1}{n\sigma}$,

This distance is called the *mean free path* for photons. Am important observation is that the Compton cross-section is inversely proportional to the mass squared of the matter particle. If we for instance consider Compton scatter of photons against protons that have a mass that is about 2000 times the electron mass, we will have a scattering cross-section that is a factor 1 / 4 000 000 smaller than the electron-photon cross-section. This has very important consequences for cosmology, as we will see later on.

Review: Check that you know and can apply the diagram rules.

Problems:

1. Draw Feynman diagrams for the annihilation process $e^+e^- \to \gamma\gamma$ (one diagram) and write down the corresponding amplitude.

2. Draw Feynman diagrams for the process $e^+e^- \to \mu^+\mu^-$ (one diagram) and write down the amplitude.

3. Draw Feynman diagrams for the elastic scattering process $e^+\mu^- \to e^+\mu^-$ (one diagram) and write down the amplitude. Note that the lepton number is conserved by family.

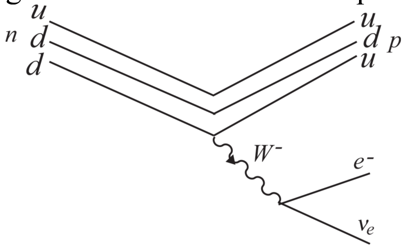
4. Show that all these reactions are forbidden and give the reason why:

$$\mu^- \to e^+e^-e^-$$
$$\mu^- \to e^-\gamma$$
$$\tau^+ \to \mu^+e^-\mu^+$$

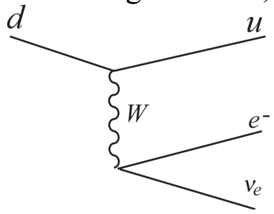
5. The Ψ meson contains a $c\bar{c}$ pair that annihilates to $\mu^+\mu^-$ via electromagnetic interaction. Draw a diagram to show how this happens.

Chapter 4. The electroweak interaction

The "usual" weak interaction is due to the exchange of the *vector bosons* W^+ and W^-. They convert a quark from the upper row in the scheme to a quark in the lower row or vice versa. Also leptons in the corresponding rows can be converted into each other. Study the figure below that shows the quark diagram of the decay of a neutron.



The *d* quark in the neutron is converted into a *u* quark that means that the charge is increased by one unit. In order for the electric charge to be conserved, a W^- has to be emitted. This W^- then decays into an electron and an anti-electron neutrino, this conserves the electron-lepton number and the electric charge. In the same way as we saw earlier there is another diagram where a positive *W* boson is emitted by the lepton pair and then absorbed by the d quark that is converted into a u quark. The sum of these two time-ordered diagrams will, as before, result in a Feynman diagram



where we haven't drawn the quarks that do not participate in the process., the so-called spectators. We have here a new type of vertex where the coupling strength in both the upper and lower vertex in this case is given by $g/\sqrt{2}$. The constant *g* is, as we will see later on, of the same order of magnitude as the charge of the electron and more precisely it is

$$g = \frac{e}{\sin\theta_W}$$

is the so-called *Weinberg angle* that has an experimental value of about 29°.

We can now write down the amplitude of the reaction by using our rules from the previous chapter

$$A \propto \left(\frac{g}{\sqrt{2}}\right)\left[\frac{1}{m_W^2c^2 - P^2}\right]\left(\frac{g}{\sqrt{2}}\right) = \left(\frac{e}{\sin\theta_W\sqrt{2}}\right)\left[\frac{1}{m_W^2c^2 - P^2}\right]\left(\frac{e}{\sin\theta_W\sqrt{2}}\right)$$

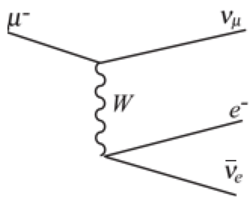
where m_W is the mass of the *W* boson that has the value 80 GeV/ c'. Incidentally, as there are two *d* quarks in the neutron that can participate, this amplitude will be multiplied by a factor of 2.

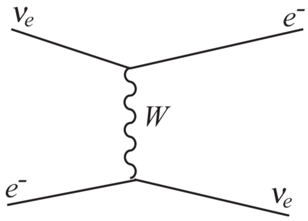
The amplitude above corresponds (see the section on scattering in your previous course in quantum mechanics) to scattering in a screened Coulomb potential

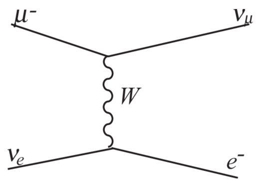
$$V(r) = \frac{1}{r}e^{-rm_Wc/\hbar}$$

The range of this potential is of the order $\frac{\hbar}{m_W c}$. We can understand this also from the uncertainty relation. If we create a particle with energy $m_W c^2$ it can exist only during the time $\Delta t \approx \hbar / m_W c^2$. Assuming that it moves with the speed of light the range of the interaction will be $c\Delta t \approx \hbar / m_W c$. If we insert the value of the mass of *W*, we find that the range is of the order 10^{-18} m. The weak interaction has an extremely *short range*.

Other similar diagrams are







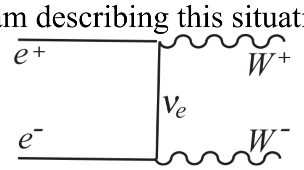
Exercises:
1. Write down the amplitudes for the reactions above. You have the same coupling as above in the respective vertex. Check that all the conservation rules are fulfilled.
2. Could the neutron decay as $n \to p\mu^-\bar{\nu}_\mu$? Why not?
3. Give another possible leptonic decay of π^- than that shown above. This alternative decay is actually the most common one.

As the mass of the *W* boson is very large, we can, if the reaction energies are small compared with the mass energy, neglect the four-momentum in the denominator of the factor that represents the inner line. In such cases we can approximate the amplitude by

$$A \propto \frac{e}{\sin\theta_W\sqrt{2}} \cdot \frac{1}{m_W^2 c^2} \cdot \frac{e}{\sin\theta_W\sqrt{2}}$$

The large mass in the denominator will make the interaction "weak" in addition to having a vary short range. This is characteristic of the weak interaction.

By colliding an electron-positron pair at high energy we can produce a W^+W^- pair. The diagram describing this situation is

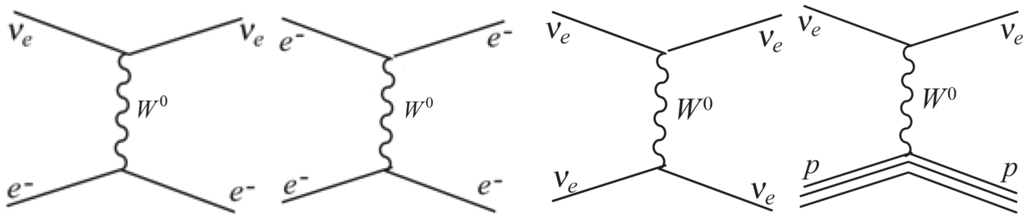


Exercise: What particles would we actually see in a track detector in this reaction?

We can write down the amplitude of this diagram. Unfortunately the probability of the process gets larger than 1 for very high energies. There must be an error in our theory. It turns out that the way of generating the weak interaction via a symmetry called local gauge invariance (we will touch upon this subject in a later chapter) also introduces a W^0 particle that is electrically neutral. The theory also predicts that W^0 will couple to e^+e^- (and $d\bar{d}$) by $-g/2$ and to $\nu\bar{\nu}$ (and $u\bar{u}$) by g/ 2. Finally also three (and four) W:s can couple together. We can then have another possible diagram for the process:

(The coupling between the three W:s has strength g.) It then turns out that the two amplitudes interfere destructively such that the final result stays reasonable.

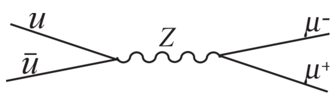
The existence of a neutral W particle makes a lot of other scattering processes possible:



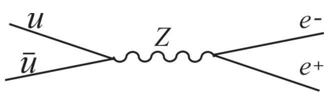
In the second of these reactions, the contribution from W^0 would be vary hard to detect in comparison to the much larger amplitude fro m photon exchange. The third process would also be very hard to detect as it involves only neutrinos that are almost impossible to detect. In the other processes it is possible to detect the presence of the neutrino by observing the recoil of the other particle.

Experimentally the effect of an electrically neutral exchange particle was observed in 1973 at CERN in νe, νp, and νn collisions. The scattering probabilities of these reactions are however not the one you would expect. It corresponds instead to the exchange of a heavier, electrically neutral particle that was called Z that had different coupling strengths.

In 1983 people succeeded in producing real Z particles in collisions between protons and antiprotons. The reactions used were



and



The mixture of weak and electromagnetic interactions

In 1968 Abdus Salam and Stephen Weinberg independently suggested a model that mixes the electromagnetic and weak interactions in a way that explains *all* the experimental observations up to the present date. They assumed that besides the W^0 there was another electrically neutral exchange particle, B. All leptons have the same probability of emitting (or absorbing) a B particle. The coupling strength is conventionally written $-g'/2$. In any scattering process where W^0 can be exchanged, it is also possible to exchange a B. These two contributions to the scattering process interfere such that it is not possible to decide which particle is exchanged. The exchanged particle will be a quantum mechanical mixture of W^0 and B. We can then write down two such possible orthogonal mixtures:

$$\gamma = B\cos\theta_W + W^0\sin\theta_W$$
$$Z = -B\sin\theta_W + W^0\cos\theta_W$$

Here θ_W is the so called *Weinberg angle*. The first combination correponds to the ordinary photon.

This is similar to the mixture of light with different polarizations. By mixing two linearly polarized electromagnetic waves we can get waves with circular polarization (left and right). In some cases the two circular polarizations propagate with different speeds in a medium that could be interpreted as if they represented different particles with different masses.

Assuming the mixture above we can write down the coupling strengths of the new particles (the mixtures) to different leptons and quarks.

ν's coupling to γ: $\frac{1}{2}\left(-g'\cos\theta_W + g\sin\theta_W\right)$
ν's coupling to *Z*: $\frac{1}{2}\left(g'\sin\theta_W + g\cos\theta_W\right)$
e's coupling to γ: $\frac{1}{2}\left(-g'\cos\theta_W - g\sin\theta_W\right)$
e's coupling to *Z*: $\frac{1}{2}\left(g'\sin\theta_W - g\cos\theta_W\right)$

We require that the neutrino does not couple to γ as it is electrically neutral. This gives

$$-g'\cos\theta_W + g\sin\theta_W = 0$$

or

$$\tan\theta_W = g'/g$$

Furthermore we require that γ couples to the electron with strength –*e*:

$$-g'\cos\theta_W - g\sin\theta_W = -e$$

Combining the results we have

$$g = e/\sin\theta_W \qquad g' = e/\cos\theta_W$$

This determines the other couplings:

νZ: $\frac{e}{2}\left(\tan\theta_W + \cot\theta_W\right)$

eZ: $\frac{e}{2}\left(\tan\theta_W - \cot\theta_W\right)$

These couplings agree well with experimental observations.

The Higgs mechanism

The mathematical theory that generates the weak and electromagnetic interaction (local gauge theory) requires that the force particles are without mass. This is obviously not the case for the weak interaction. Also we would like to explain why the mixtures we see in Nature are precisely those we have above. Now, a massless particle moving in a medium can behave as if it had a mass. Compare for instance with an electromagnetic wave moving in an electron plasma with number density n_e, which we treated in the earlier course (FYS 022). In that case, the electromagnetic waves (photons) behave as if they had a mass determined by the *plasma frequency* ω_p 0)

$$\left(mc^2\right)^2 = \left(\hbar\omega_p\right)^2 = \hbar^2 \frac{n_e e^2}{\varepsilon_0 m_e} \propto n_e e^2$$

We can interpret this as if the photon is repeatedly absorbed and reemitted and thereby is slowed down to behave like a massive particle.

We see that the mass squared will be proportional to the coupling strength to the medium and to the number density of the medium particles. Now we assume that vacuum is a special plasma, the *Higgs medium*, consisting of neutral, spinless, neutrino-like particles and their anti-particles that we call *N* and $\bar{N}$. We assume that *W*, *Z* and γ interact in the same way with these particles as with ordinary neutrinos. Then *W* will interact with the *N* particles, creating an intermediary positive particle to get a mass given by

$$m_W^2 = K\left(\frac{g}{\sqrt{2}}\right)^2 = K\frac{g^2}{2}$$

A *Z* particle will get a mass

$$m_Z^2 = 2K\left[\tfrac{1}{2}\left(g'\sin\theta_W + g\cos\theta_W\right)\right]^2 = K\frac{g^2}{4\cos^2\theta_W}$$

The factor 2 in front of *K* comes because *Z*, being neutral, can interact with both *N* and $\bar{N}$.

This implies that we can express the mass of *Z* by the mass of *W*:

$$m_Z = \frac{m_W}{\cos\theta_W} = 91\,\mathrm{GeV/c^2}$$

This fits very well with the experimental value of the *Z* mass and gives an independent confirmation of the theory.

The *photon* gets mass zero as it doesn't couple to the neutrinos and thus not to the Higgs medium.

We can investigate what happens if a W^0 or a *B* is absorbed by the *N* particles in the Higgs medium. As W^0 and *B* are neutral, the *N* particles will continue as (virtual) *N* particles. When the *N* particles again emit W^0 and *B* they will not remember what they absorbed earlier. The amount of W^0 and *B* will be determined by the respective coupling strengths and the emitted mixture will be

$$W^0\frac{g}{2} - B\frac{g'}{2} = W^0\frac{e}{2\sin\theta_W} - B\frac{e}{2\cos\theta_W} =$$

$$\frac{e}{2\sin\theta_W\cos\theta_W}\left(W^0\cos\theta_W - B\sin\theta_W\right) = \frac{e}{2\sin\theta_W\cos\theta_W}Z$$

Whatever combination of W^0 and *B* that was absorbed there will always be a *Z* emitted! The Higgs medium will act as a filter that separates any mixture of W^0 and *B* into a *Z* that interacts with the medium and gets mass, and a γ that will not interact with the medium and thus will stay massless.

We can therefore say that

massless exchange particles + Higgs medium
= Some of the exchange particles get mass

We cannot observe the particles in the Higgs medium directly, only through the fact that they give some of the exchange particles mass. However, in complete analogy to an electrically charged plasma, there can be longitudinal pressure waves in the plasma. These waves are quantized and would correspond to a particle, the *Higgs particle* that should be observable. The experimental difficulties were formidable, but in 2013 the Higgs particle was finally found at CERN with a mass of about 120 GeV.

Couplings to the quarks

W couples to the quarks in the same way as to the leptons, i.e. $W^{\pm}$ couples with strength $g/\sqrt{2}$ and W^0 couples with strength g / 2 to the upper row, u, c, t and with $-g$ / 2 to the lower quarks: d, s, b. However, the B particle couples with strength g' / 6 to all the quarks. This gives correct electric charges to the quarks.

Exercise: Check that with these couplings the photon couples with strength $+2e$ / 3 to the upper quarks and with strength $-e$ / 3 to the lower quarks. Then show that the quark couplings to Z are:

Upper quarks: $-\frac{g'}{6}\sin\theta_W + \frac{g}{2}\cos\theta_W = \frac{e}{2}\left(-\frac{1}{3}\tan\theta_W + \cot\theta_W\right)$

Lower quarks: $-\frac{g'}{6}\sin\theta_W - \frac{g}{2}\cos\theta_W = -\frac{e}{2}\left(\frac{1}{3}\tan\theta_W + \cot\theta_W\right)$

These couplings describe the experiments very well, again a confirmation of the Salam-Weinberg model.

Some complications in the electro-weak model

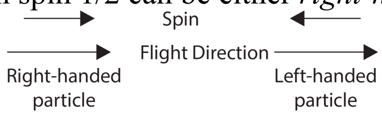
a) It turns out that we have to modify our original scheme for the quarks somewhat. For the two first families we have instead

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \text{with} \quad \begin{aligned} d' &= d\cos\theta_C + s\sin\theta_C \\ s' &= -d\sin\theta_C + s\cos\theta_C \end{aligned}$$

θ_C is the *Cabbibo angle* that has an experimental value of about 13°. The weak interaction will then change a d' quark into a u quark. As d' is a mixture of d and s this means that $d \leftrightarrow u$ with probability $\cos^2\theta_C$ or 0.95 and $s \leftrightarrow u$ with probability $\sin^2\theta_C$ or 0.05. This means that there is a small probability that the weak interaction converts quarks between the families. There is actually a mixing between all the families that allows also conversions between the c- and t families and also with very small probability between the t and u families. Note, however, that conversions between families will always go in a zigzag manner, never horizontally between families. It is not possible for example to convert a s quark to a d quark with the help of a Z.

Exercise: Show that given that the vertex dsZ has zero coupling, this implies zero coupling for $d's'Z$.

b) A particle with spin 1/2 can be either *right-handed* or *left-handed*:



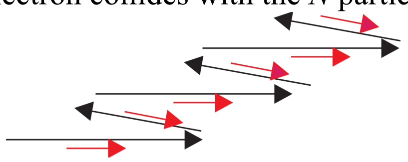
For massless particles (for instance neutrinos), that move with the speed of light, the handedness is a given property that cannot change. It turns out that *neutrinos* are *left-handed* in nature while *anti-neutrinos* are *right-handed*. We also speak of positive and negative helicity for right- and left-handedness respectively.

For a massive particle we can change the handedness by looking at the particle from a frame that moves parallel with the particle and with a higher speed. The particle will then seem to move in the opposite direction while the spin direction is unchanged.

All couplings that we have given earlier are only valid for left-handed particles or right-handed antiparticles. All couplings with W (W^+, W^-, W^0) and right-handed particle / left-handed anti-particles are zero! This leads to some modifications in our earlier scheme for couplings that you can study in the attached sheet with couplings at the end of this book. In most problems you are allowed to do the computations as if all the particles are left-handed (right-handed anti-particles) i. e. you can use the simple theory given before.

(We don't know if right-handed neutrinos / left-handed antineutrinos exist in nature. If they exist, they will not interact with the W:s. As the neutrino is neutral electrically it will not interact with γ. But then it cannot interact with B either. Neutrinos do not interact strongly. This means that such neutrinos would only interact gravitationally which means that they would be extremely hard (at present impossible) to detect experimentally.)

c) The Higgs medium is also used to give the electron and the quarks their masses, that is they are fundamentally massless. In fact this also explains why the electron can be both right- and left-handed. The massless electron has a definite handedness. When it interacts with the Higgs medium it is slowed down and we can think about this as if the electron collides with the N particles and moves in zig-zag in space, see figure.

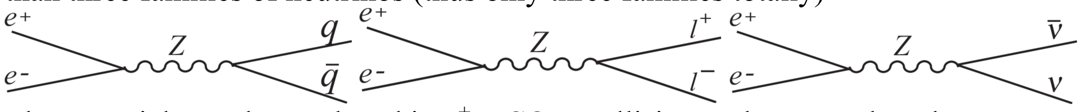


The small red arrows show the spin direction. The spin direction will not change in the collisions that means that if the electron is originally right-handed it will during the time it moves ”backwards” be left-handed and during this time has possibility to interact with the W particles. An electron at rest will be left- and right-handed with equal probability.

This gives an explanation why the π-meson prefers to decay to a mu (and a μ anti-neutrino) instead of to an electron (and an electron anti-neutrino). The π-meson has spin zero. The outgoing anti-neutrino is right-handed. By angular momentum conservation the outgoing lepton then also has to be right-handed. But the intermediary vector boson does not couple to right-handed leptons. However, as shown above, a lepton moving in the Higgs medium is partly left-handed. The muon is about 200 times more massive than the electron and so moves much slower and therefore has a much larger left-handed component making it couple much stronger to the vector boson.

An interesting final example

We are now in a position that we can show quite convincingly that there are no more than three families of neutrinos (thus only three families totally)



The Z particle can be produced in e^+e^- COM collisions. The Z can then decay to hadron, that is a $q\bar{q}$ pair, a charged lepton pair or a neutrino pair. For the $q\bar{q}$ pair only $u\bar{u}$, $c\bar{c}$, $d\bar{d}$, $s\bar{s}$, and $b\bar{b}$ are possible energetically. For the lepton decays e^+e^-, $\mu^+\mu^-$, and $\tau^+\tau^-$, are possible. In the diagrams above the left vertex is the same and can be left out. Also if the experiments are done for energies in the neighbourhood of the Z rest mass we can neglect photon exchange. The energies of the decay particles (=45 GeV) are high enough to make these particles essentially massless. However, we have to include the possibility that the particle pair can be either left- or right-handed. We calculate the (relative) probabilities using a Weinberg angle of 28.75°.

Left-handed decays:

2

Z to upper $q\bar{q}$ pair $\left[\frac{e}{2}\left(-\frac{1}{3}\tan\theta_W + \cot\theta_W\right)\right]^2 = e^2 \cdot 0.672$

Z to lower $q\bar{q}$ pair $\left[-\frac{e}{2}\left(\frac{1}{3}\tan\theta_W + \cot\theta_W\right)\right]^2 = e^2 \cdot 1.006$

Z to charged lepton pair $l\bar{l}$ $\left[\frac{e}{2}\left(\tan\theta_W - \cot\theta_W\right)\right]^2 = e^2 \cdot 0.406$

Z to neutrino pair $\nu\bar{\nu}$ $\left[\frac{e}{2}\left(\tan\theta_W + \cot\theta_W\right)\right]^2 = e^2 \cdot 1.406$

Right-handed decays:

Z to upper $q\bar{q}$ $\left[-\frac{2e}{3}\tan\theta_W\right]^2 = e^2 \cdot 0.134$

Z to lower $q\bar{q}$ pair $\left[\frac{e}{3}\tan\theta_W\right]^2 = e^2 \cdot 0.033$

Z to lepton pair $l\bar{l}$ $\left[e\tan\theta_W\right]^2 = e^2 \cdot 0.301$

Z to neutrino pair $\nu\bar{\nu}$ 0

Summing the left- and right-handed alternatives we get

Z to upper $q\bar{q}$ pair $e^2 \cdot 0.806$

Z to lower $q\bar{q}$ pair $e^2 \cdot 1.039$

Z to lepton pair $e^2 \cdot 0.707$
Z to neutrino pair $e^2 \cdot 1.406$
Then
$P(Z \to q\overline{q}) \propto e^2 \cdot 3(2 \cdot 0.806 + 3 \cdot 1.039) = e^2 \cdot 14.188$ (The first factor 3 is for the three quark colours, see chapter 5)
$P(Z \to l^+l^-) \propto e^2(0.406 + 0.301) = e^2 \cdot 0.707$ (per generation)
$P(Z \to \nu\overline{\nu}) \propto e^2 \cdot 1.406$ (per generation)

The experimental total width of $Z \to$ anything is 2.490 GeV.
The experimental width of $Z \to$ hadrons is 1.741 GeV.
The experimental width of $Z \to$ charged leptons is 0.0838 GeV.
The computed width of $Z \to$ neutrinos normalized with the hadron width
1.741 Gev·1.406/14.188 =0.173 GeV.
The computed width of $Z \to$ neutrinos normalized with the lepton width
0.0838 Gev·1.406/0.707 =0.167 GeV.
Average 0.167 GeV.
We now make a check $\left(\dfrac{Z \to hadrons}{Z \to leptons}\right)_{\text{exp}} = 20.77$ and $\left(\dfrac{Z \to hadrons}{Z \to leptons}\right)_{\text{theor}} = 20.07$
Assuming N families of neutrinos we have that
N·width of $Z \to neutrinos =$
Total width – (width of $Z \to hadrons$) – 3·(width of $Z \to leptons$)
(We have 3 kinds of possible lepton pairs)
Inserting numbers we have
$N \cdot 0.170 = 2.249 - 1.741 - 3 \cdot 0.0838 = 0.4976$
Finally giving $N = 2.93$!!!, i.e. there are three families.

Review: Repeat the mechanism behind the weak interaction. Why is it weak? Why is the range so short? Why is the W^0 needed? Explain the mixing between the weak and electromagnetic interaction. Describe how we fix the coupling constants. How do the weak gauge particles get mass? How could we find a relation between the W and Z masses. What is the Cabbibo angle? How is the theory changed if we have right-handed particles?

Problems:
1. The Ψ^* meson has the quark content $c\overline{c}$ and mass 3.77 GeV/c^2. It can decay in a D^+D^- pair with quark content $c\overline{d}$ and $\overline{c}d$ respectively. The c quark can decay to $s + u + \overline{d}$.
a) Which particle is involved in the decay of the c quarks? Show in a diagram what happens.
b) Give some possible final states for the decay of the D^+ meson.
2. By comparing the expression for the Rutherford cross-section in the Born approximation (see the earlier course in quantum mechanics) and the expression you get to first order in our simplified model of quantum electrodynamics for e^+e^- scattering We can determine the proportionality constants that we have neglected. We find that the cross-section can be written

$$\frac{d\sigma}{d\Omega}=\left(\frac{m}{2\pi}\right)^2\left[\frac{e^2/\varepsilon_0}{(P_1-P_1')^2}\right]^2$$

where $(P_1-P_1')^2$ is a four-vector scalar product with P_1 and P_1' being the four-momenta of the in- and outgoing electron respectively. *m* is the electron mass. The expression above is valid for non-relativistic particles.

Now repeat the steps of how to compute the density of states for non-relativistic particles! Fundamental in this computation is.:

a) The number of states in an infinitesimal "volume element" in p-space (momentum) that is given by

$$\frac{V}{(2\pi)^3}\frac{1}{\hbar^3}dp_x dp_y dp_z=\frac{V}{(2\pi)^3}\frac{1}{\hbar^3}p^2dpd\Omega_p$$

V is the normalisation volume. As we know that it anyhow will drop out in the final result we put it equal to 1 for the moment.

b) To get the density of states in energy (that is what we need in Fermi's Golden Rule), we have to translate the momentum in the expression to energy. For a non-relativistic particle we do this by using the relation $p^2=2mE$. Now compute the density of states in energy for such a particle!

c) For a massless relativistic particle (photon, neutrino) we have instead $p=E/c$. Compute the density of states in this case.

d) For a relativistic massive particle we have $p^2=E^2/c^2-m^2c^2$. Compute the density of states also for this case!

e) In the cross section we divide by the influx of particles (the number of incoming particles / time and area). This influx of particle is proportional to the speed of the particles divided by the normalisation volume. Determine the influx of particles expressed in *m*, *p*, and *E* and the normalisation volume *V* for massive non-relativistic particles, massless particles and relativistic massive particles respectively.

f) Show that for relativistic particles (massive or massless) you get the same expression for the cross section as above but with the substitution $m\to E/c^2$.

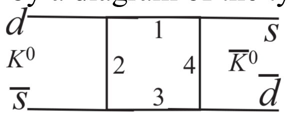
g) Many neutrinos are produced in the sun in the fusion of hydrogen and reach the earth with the enormous flux $6\cdot 10^{14}$ m^{-2}s^{-1}. Estimate the probability per second that a neutrino that passes your body from head to toes will interact with you. Assume that the reaction that happens is

$$\nu_e+d\to e^-+u$$

Furthermore, assume that the neutrinos have an energy of about 0.5 MeV. Use the cross-section from f) above, but change the electron charge (the electromagnetic coupling) with the coupling you have with a *W*-exchange. Finally you must modify the denominator in the factor from the internal line due to the mass of *W*. You can also, if you want, use that $\alpha = \dfrac{e^2}{4\pi\varepsilon_0\hbar} \approx \dfrac{1}{137}$. Hint: Use that the reaction probability per time = cross-section · influx. Answer: 10^{-4} s^{-1}.

3. Estimate the lifetime of the decay $\pi^- \to \mu^- \bar{\nu}_\mu$. Use the cross section in problem 2, assume that the quarks move with the speed of light in a "blob" with radius 1 fm and collide with each other. The lifetime = 1 / the reaction probability. Answer: 10^{-9} s.

4. Via the weak interaction K^0 can transform into its own antiparticle $\bar{K}^0$. This can occur by a diagram of the type



What particles can substitute for 1, 2, 3, and 4 in the figure (several possibilities)? Why will the probability for this conversion be very small? (Two reasons!)

5. If a *c* quark is produced it can be part of a $\Lambda_c = (udc)$ baryon. Give some possible decays of this particle, with leptons and without leptons in the final state. Draw diagrams! State how these decays are influenced by the Cabbibo angle.

6. We study the ratio between cross sections of two neutrino reactions (at the same energy)

$$R = \frac{P(\nu_\mu + N \to \nu_\mu + X)}{P(\nu_\mu + N \to \mu^- + X')}$$

N is here an atomic nucleus, *X* and *X'* respectively the resulting nucleus in the final state. Assume that the reaction energy is much smaller than the mass of *W* and *Z* but large enough for the muon mass to be neglected. Furthermore, assume that the nucleus contains as many protons as neutrons. If you want you may also assume that all quarks are left-handed. Compute using the Standard Model the value of *R*.

7. A *u*-quark can, via the weak interaction, change into a superposition *d'*, of *d*- and *s*-quarks:

$$d' = d\cos^2\theta_C + s\sin^2\theta_C$$

The Cabbibo angle, θ_C, can be determined by studying the decay rates of the decays $\pi^+ \to \mu^+\nu_\mu$ and $K^+ \to \mu^+\nu_\mu$ The decay rate Γ is defined by the relation $\Gamma = B/\tau$ where τ is the mean lifetime and *B* the branching ratio. Γ is proportional to a kinematical factor (essentially the density of states · factors that appear when you sum over the spin of the outgoing particles) and a dynamical factor according to

$$\Gamma \propto \frac{m_l^2\left(m_m^2 - m_l^2\right)}{m_m^2} \cdot \frac{1}{m_W^4} \cdot \left|coupling\ strength\ q_1\bar{q}_2 \to W\right|^2$$

where m_m is the meson mass and m_l, the lepton mass (in this case the muon mass).

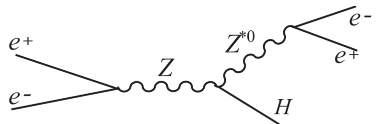
Estimate the Cabbibo angle using the ratio $\dfrac{\Gamma(K^+ \to \mu^+\nu_\mu)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)}$

(The value will only be approximate as it is also influenced somewhat by the different binding of the quarks in the K- and π meson).

Experimentally we know for the decays
$$K^+ \to \mu^+ \nu_\mu : \tau = 1.24 \cdot 10^{-8}\,\mathrm{s};\ B = 0.635$$
$$\pi^+ \to \mu^+ \nu_\mu : \tau = 2.64 \cdot 10^{-8}\,\mathrm{s};\ B = 1.00$$

Answer: 15°.

8. At the LEP accelerator in CERN in Geneva, electrons collide with positrons with a total energy in the centre of mass system of 91 GeV, corresponding to the mass of Z^0. Among other things people search for the Higgs particle that would be produced by the reaction



where Z^{*0} is virtual (an intermediate state that doesn't have to have the correct mass) while Z^0 is real and H is the Higgs particle.

a) What is (today very unrealistically) the mass of the Higgs particle if the outgoing electron and positron have energies 15 GeV and 25 GeV respectively and the angle between their momenta is 90°? (42 GeV/c^2)

A Higgs particle with this mass would decay predominantly in $b\bar{b}$ pairs. These pairs will create B- and $\bar{B}$-mesons respectively. The b quark decays into a c quark that in turn decays.

b) Write down some characteristic decays of the B^- meson (that has the quark content $b\bar{u}$). Show that these decays often generate K-mesons and leptons.

c) The Z^0-particles that are produced at LEP will often decay in a quark-antiquark pair. What quark types will be most abundant? (Hint: Look at the difference in coupling between up and down quarks Z^0.)

d) If a c-quark is produced it can end up in a $\Lambda_c = (udc)$. Draw some quark diagrams (with or without leptons in the final state) that show possible decays of Λ_c.

e) Indicate how the decays in d) are influenced by the Cabibbo angle.

f) Λ_c can decay weakly by $\Lambda_c \to \Lambda^0 \mu^+ \nu_\mu$. In one experiment a Λ_c decays at rest according to this reaction. The momenta of Λ^0 and μ^+ was measured to 496 MeV / c and 533 MeV/ c respectively. Compute the mass of Λ_c if the angle between the detected particles Λ^0 and μ^+ was 123°. m_Λ = 1115 MeV/c’, m_μ = 105.7 MeV/c’. (2256 MeV/c^2)

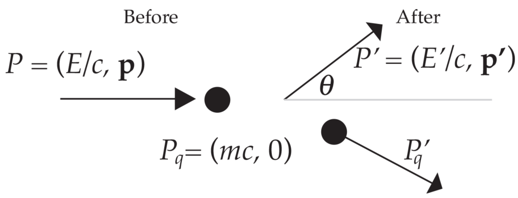
Chapter 5. Quantum Chromodynamics (QCD)

The idea that the hadrons are composed of quarks is quite nice. However, there are several problems with this quark model:

• Where are the free quarks?
• Why do quarks combine only in triplets or quark-antiquark combinations?
•There are problems with the simple quark model and the Pauli principle.

If we start with the first problem, it is actually possible to see bound quarks in the nucleons. Scattering experiments are often used to investigate the finer structure of a system. This was the idea when Ernest Rutherford scattered alpha particles against gold atoms in 1911 that led to the picture that t-here is an almost point-like nucleus in an atom.

Suppose now that we want to explore the structure of a proton by scattering electrons against it. The size of a proton is of the order of 1 fm that means that we need electrons with a wavelength that is smaller than this size. Such electrons have an energy of the order of 100 MeV and are highly relativistic, meaning that we can neglect their rest mass (0.5 MeV) and threat them as massless and photon-like. We assume that the electrons are scattered against some more or less point-like objects (quarks?) inside the proton. To first approximation we assume that these quarks are at rest. The relativistic calculation of such a collision is precisely that of Compton scattering:



Energy-momentum conservation gives

$$P + P_q = P' + P_q' \Rightarrow P^2 + P'^2 + P_q^2 - 2PP' + 2\left(P - P'\right) = P_q'^2$$

Neglecting the mass of the electron we have

$$P_q\left(P - P'\right) = PP'$$

or

$$m\left(E - E'\right) = EE'/c^2 - pp'\cos\theta$$

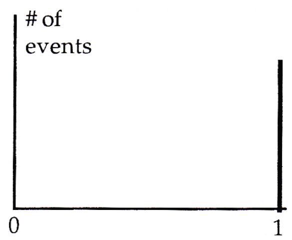
or

$$x = \frac{mc^2}{m_p c^2} = \frac{EE'\left(1 - \cos\theta\right)}{m_p c^2\left(E - E'\right)}$$

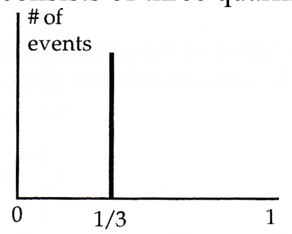
where we have normalised the result by dividing by the rest energy of the proton. Knowing E and measuring E' and θ makes it possible to calculate x, essentially the mass with which the incoming electron collided.

We now measure E' for fixed θ and E and get a distribution $F(x)$. We can then change the values of E and θ and repeat the measurements.

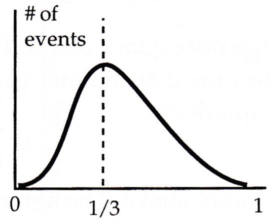
Now if the proton is elementary we would expect the following result of such an experiment:



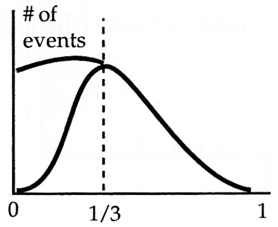
If the proton consists of three quarks at rest we would expect:



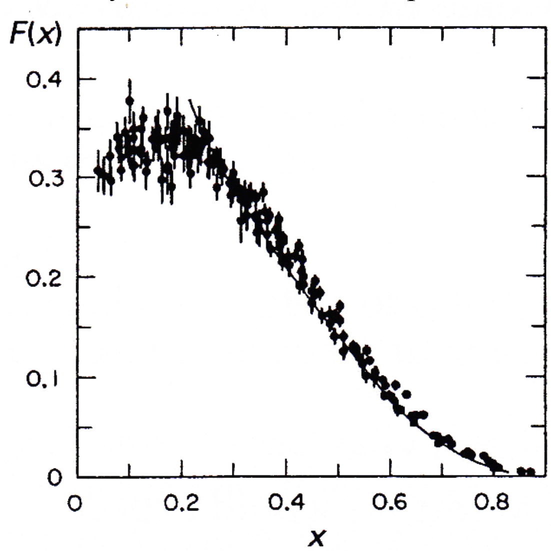
If we take into consideration that the quarks can move inside the proton our calculation is not valid but we could guess that we would have a continuous distribution around $x = 1/3$:



Finally if we also take into consideration that there are virtual quark-antiquarks that can exist for short times inside the proton, so called sea quarks, the distribution would look something like this:



This is actually what we see in the experiments!! I The figure below shows this:



All the points lie on a common curve that is an indication of the existence of quarks. It is even possible to investigate the different kinds of quarks inside the proton. Study the reactions

$$\nu_e + p \to e^- + hadrons$$

$$\bar{\nu}_e + p \to e^+ + hadrons$$

Exercise: These reactions involve the exchange of a *W*-boson. Show that in the first reaction only the *d*-quark in the proton can interact. Show that in the second reaction only a *u*-quark can interact. This means that the neutrinos only "see" a *d*- and a *u*-quark respectively.

From these scattering experiments we can determine the pure quark distributions $F_u(x)$ and $F_d(x)$. The total function $F(x)$ will be the sum of the pure distributions each weighted with the square of the electric charge of the respective quark or

$$F(x) = \frac{4}{9}F_u(x) + \frac{1}{9}F_d(x)$$

The result of this operation is shown as the line in the figure above. The agreement with the results from electron scattering is a strong support for the quark theory and also confirms the value of the electric charges of the quarks.

We get yet another interesting picture if we look at the differential cross-section for e^+p and e^-p scattering. More specifically we look at the two "neutral current" reactions (the exchanges particle is neutral, a photon and/or a *Z*)

$$e^+p \to e^+X$$

$$e^-p \to e^-X$$

and the two "charged current" (*W*-exchange) reactions

$$e^+p \to \bar{\nu}_e X$$

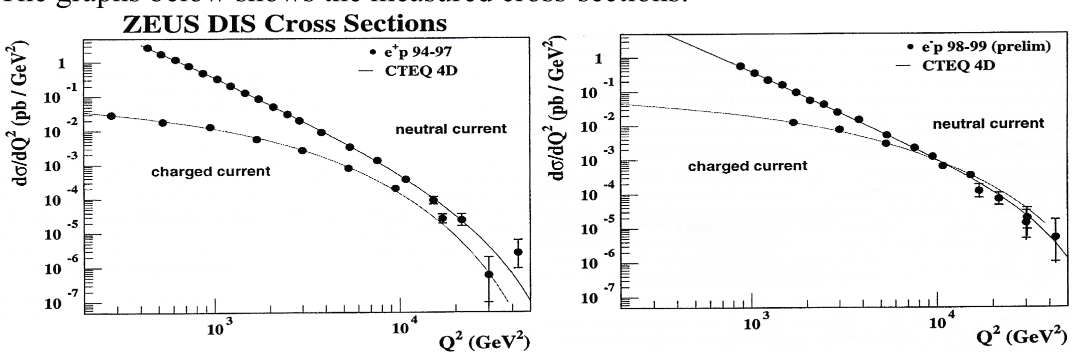
$$e^-p \to \nu_e X$$

In all these reactions the exchanged particle reacts with a quark in the proton. What is measured is the differential cross-section

$$\frac{d\sigma}{dQ^2} \propto \frac{d\sigma}{d\Omega} \propto \left(\frac{coupling\ factors}{M^2c^2 - \left(P_{e,in} - P_{e,out}\right)^2} \right)^2 = \left(\frac{coupling\ factors}{M^2c^2 + Q^2} \right)^2$$

where M is the mass and $P_{e,in} - P_{e,out}$ the four-momentum of the exchanged boson.

The graphs below shows the measured cross-sections:



We can see several interesting things in these diagrams.

a) The slope of the neutral cross-section for reasonably small Q^2 is about -4 in the logarithmic scale. (Explain why!)

b) We see that at high Q^2 the charged and neutral cross-sections become more or less the same.(Explain why!)

c) The charged cross-section for e^-p for reasonably small Q^2 is roughly twice the cross-section for e^+p.(Explain why!)
d) Why does the neutral current curve bends for higher Q^2 (Explain why!)

So experiments tell us there are quarks. But why don't we see any free quarks? It turns out that we can solve all the problems that we listed in the beginning of this chapter by solving the problem with the Pauli principle. Study the Δ^{++} hadron that has the quark content

$$\Delta^{++} = \left(u\uparrow u\uparrow u\uparrow\right)$$

The Δ^{++} has spin 3/2 and all the quarks must have parallel spins. Thus we have three identical fermions in the same state. This violates the Pauli principle. One way out of this dilemma is to postulate that the quarks are not identical because the have different "colour charge". We call conventionally these colour charges *red* (R), *green* (G) and *blue* (B). The particle above will then look like

$$\Delta^{++} = \left(u_R\uparrow u_G\uparrow u_B\uparrow\right)$$

The particles are no longer identical and we have solved the conflict with the Pauli principle.

This seems *ad hoc* but it turns out that this trick also explains several other properties of the strong interaction:

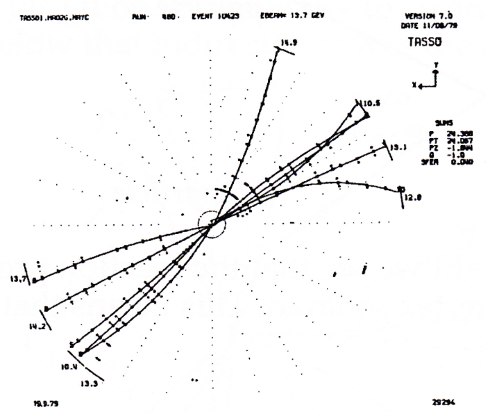
• A gauge theory similar to the one that generates the weak and electromagnetic interaction but using three colour charges generates 8 force particles (*gluons*) in the strong interaction. Contrary to the photon the gluons are charged with colour. This means that gluons interact with other gluons! We can have bound states with gluons and as the gluons are massless it will be energetically favourable to fill empty space with as many as possible of such bound gluons. Vacuum will be a soup of gluons! As a matter of fact also the weak force particles interact with each other. The difference is here that as the *W*:s have mass the bound states will not as with the gluons have negative energy.

• When we separate two different *electric* charges it is energetically favourable for the electric field to spread in the space between the charges. With the strong interaction it is just the opposite! The colour field tries to use as little space as possible due to the vacuum- gluon-soup. This results in the field-lines being compressed into a one-dimensional tube or *string* between the quarks. In such a case, the force between the quarks will be *constant*, independent on the distance between the quarks! This in turn implies that we cannot separate the quarks, to do this we need an *infinite* energy. This explains why we don't see free quarks!

• The infinite energy in a free colour field means that only quark combinations where there are no external colour field lines will be possible. Allowed quark combinations will be quark-antiquark, quark-quark-quark or antiquark-antiquark-antiquark that is exactly the combinations that we see in nature!

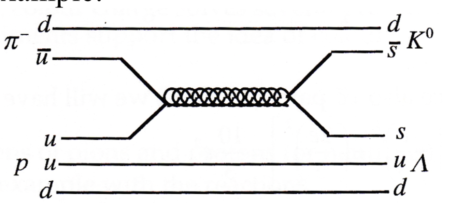
• In a nucleon the colour field will because of this extend only very little outside the nucleon. This results in the strong force having a very short range.

• When we separate two quarks in a collision process, the colour field between them will be drawn out into a kind of elastic string. When the energy stored in the field becomes large enough to create a new particle (a meson), the string will break while creating a quark-antiquark pair. This can happen several times and experimentally we see this as a shower of mesons, essentially moving in the same direction as the knocked-out quark. This is called a *jet* and is readily seen experimentally.

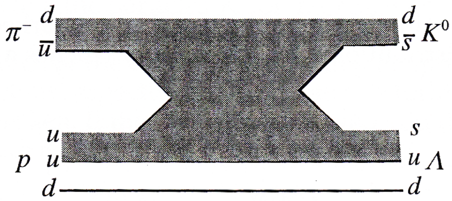


• By studying in the ratio between the probability to produce hadrons and a muon-anti-muon pair in an electron-positron annihilation we can check the theory of the strong interaction, especially the concept of colour charge. The experiments confirm the theory. We will look into this in the next section.

Exactly as for the electromagnetic and weak interaction we can draw diagrams of scattering processes and decays. A gluon exchange is then visualised by a spiral line. One example:

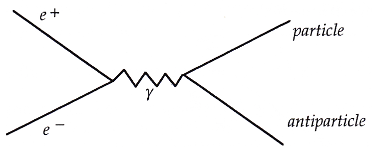


Strictly the probability of exchanging gluons increases with the number of exchanged gluons. Thus, a more correct way of drawing this in the diagram would be to draw an "exchange surface":



Electron-positron annihilation

Interesting experiments can be done by colliding electrons and positrons at high energy in their centre of mass system. The electron and positron, being particle and antiparticle, annihilate and create a blob of energy. This blob of energy can then create a new particle-antiparticle pair of any kind. If the energy is not too big (less than the mass of the *Z* particle) the only diagram that can contribute essentially is



If we just study the vertex to the right we see that the reaction probability will be proportional to the charge in this vertex squared. This means that the probability of producing a $\mu^+\mu^-$ pair will be

$$P\left(e^+e^- \to \mu^+\mu^-\right) \propto e^2$$

On the other hand the probability of producing hadrons will be

$$P\left(e^+e^- \to q\overline{q}\right) \propto \sum Q_q^2$$

Assume now that the COM energy is such that only the lightest quarks *u*, *d*, and *s* can be created. For each of them the created pair can be red-antired, green-antigreen, blue-anti-blue, 3 possibilities which gives

$$\sum Q_q^2 = \left[\left(\frac{2e}{3}\right)^2 + \left(-\frac{e}{3}\right)^2 + \left(-\frac{e}{3}\right)^2\right] = 2e^2$$

This gives

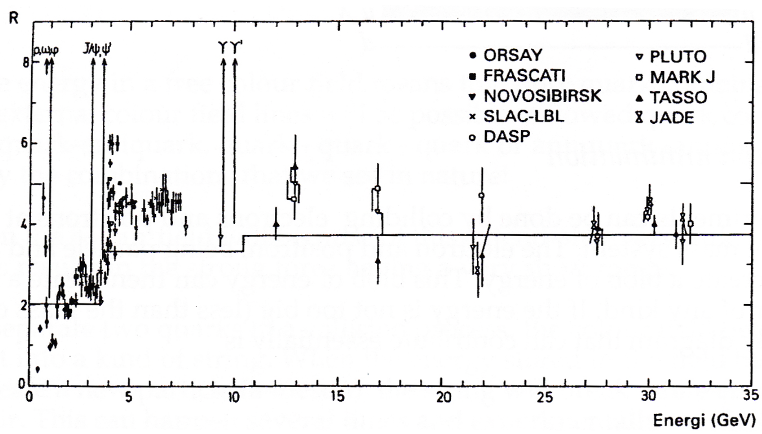
$$R = \frac{P\left(e^+e^- \to q\overline{q}\right)}{P\left(e^+e^- \to \mu^+\mu^-\right)} = 2$$

If the energy is high enough to produce also $c\overline{c}$ pairs (3.7 GeV) we will have

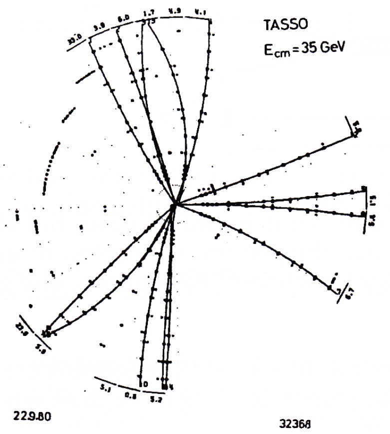
$$P\left(e^+e^- \to q\overline{q}\right) \propto \sum Q_q^2 = \left[\left(\frac{2e}{3}\right)^2 + \left(-\frac{e}{3}\right)^2 + \left(-\frac{e}{3}\right)^2 + \left(\frac{2e}{3}\right)^2\right] = \frac{10}{3}e^2$$

and $R = 3\frac{1}{3}$.

Another threshold is reached at 10.5 GeV where pairs of bottom quarks can be produced and we have $R = 3\frac{2}{3}$. The figure shows experimental results with the theoretical prediction marked as a line. There are peaks in the experimental distribution when the energy reaches a threshold for producing a bound $q\overline{q}$ pair.



The quantum chromodynamics also predicts that it would be possible to produce a gluon in an e^+e^- interaction. This gluon will give rise to an additional jet. A typical such event is shown in the figure below that indirectly proves the existence of gluons.



Review:
What experiments indicate that there are quarks? Why don't we see free quarks? Introducing colour charge solves several problems with the strong interaction, Which? What experiments support the idea of colour?

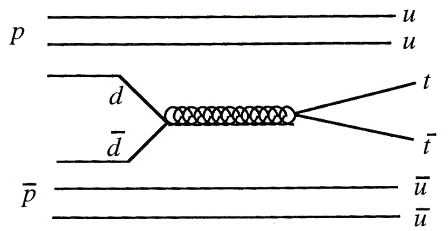
Problems:
1. In collisions of pions and protons (or neutrons) you can easily produce the mesons K^+ and K^0 for example with the reactions

$$\pi^- p \to K^0 \Lambda^0$$

a) Draw a quark diagram showing this.
b) It is much more difficult to create K^- and $\overline{K}^0$ Explain why!
c) In a similar way not all K-mesons interact easily with nucleons and are absorbed in matter. Which of the particles K^+ or K^- reacts most easy?

2. Decide which of the reactions below that are possible by strong, electromagnetic and weak interaction respectively. If a reaction is forbidden you must state the conservation law that is broken. If the reaction is allowed you draw a quark diagram. In some of the reactions there is more than one possibility.
a) $\pi^- p \to \pi^0 n$ b) $K^- p \to \Xi^0 K^0$ c) $p \to n e^+ \nu_e$
d) $p \to e^+ \pi^0$ e) $\Omega^- \to \Lambda^0 K^- K^0$ f) $\pi^- p \to K^+ \Sigma^-$
g) $\pi^- p \to K^- \Sigma^+$ h) $K^0 \to \pi^+ \pi^-$ i) $K^0 \to \pi^+ \pi^- \pi^0$
k) $K^0 \to \pi^+ \pi^- \pi^0 \pi^0$ l) $\pi^0 \to \gamma\gamma$ m) $\pi^+ n \to K^0 \Sigma^+$
n) $K^- p \to \Xi^- K^+$ o) $\Lambda^0 \to \pi^+ e^- \overline{\nu}_e$ p) $K^+ \to \pi^0 \mu^+ \nu_\mu$

3. One of the world's largest particle accelerators, the Tevatron, is situated at the research centre Fermilab outside Chicago. Here protons and antiprotons are accelerated to an energy of 900 GeV and collide with a centre of mass energy of 1.8 TeV. In 1994 scientists at Fermilab could, for the first time, directly verify the existence of the top quark, *t*. Its mass was found to be 175 GeV/c’ and its lifetime $4 \cdot 10^{-25}$ s. The main mechanism for the production of the top quark is annihilation of one quark from the proton with an antiquark from the antiproton like for example in the following diagram:



Now, the proton and antiproton don't contain only quarks, but also a large number of gluons that together carry about half the proton / antiproton momentum. A top quark then can also be produced by the interaction of two gluons, one gluon from each of the proton and antiproton.

a) Draw the two Feynman graphs that to lowest order contribute to this process $gg \to \bar{u}$. Hint: The gluons can couple to each other, leading to vertices where three gluons meet.

b) Write down the expressions for the respective amplitudes. Neglect coupling strengths, spin and colour factors as well as kinematical factors in the expression, concentrate on the denominator.

Top quarks decay according to $t \to bW^+$ before they have time to be bound in a hadron. The way to "see" a top quark is by reconstructing its energy and momentum from its decay products.

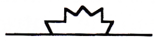
c) At the Tevatron sometimes top quark pairs are produced by weak interaction. Draw a Feynman diagram for such a reaction and write down the expression for the reaction probability. Again neglect spin, colour and kinematical factors, but this time include the couplings.

d) The *b*-quark from the decay of the top decay sometimes ends up in a B^0 meson (with quark content $b\bar{d}$). Give examples of different final states of the decay of this meson both with and without leptons in the decay products.

4. Certain measurements of the ratio between the cross sections of the reactions $e^+e^- \to b\bar{b}$ and $e^+e^- \to \mu^+\mu^-$ have shown a small discrepancy from the value that is predicted by the Standard Model. Compute the value predicted by the Standard Model. Assume that the energy is high enough such that only *Z* exchange is allowed. Also assume only left-handed couplings. Hint: Colour!
Answer: 7.6 (Including right-handed couplings results in the value 4.4.)

*Chapter 6. Renormalisation and gauge symmetries

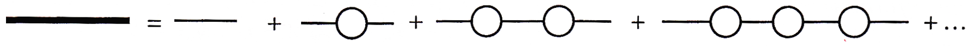
Let us now look at an exchanged electron which we have up till now represented by a line. This electron can of course (and does) emit a photon that can then be absorbed. Symbolically



If we correctly calculate the contribution from this Feynman diagram it turns out that it is infinite. And the situation is actually worse. We can have diagrams of the type

a b c

that all give infinite contributions. Even worse, each photon in the photon line can for a moment be transformed into an electron-positron pair where the electron and positron in turn can emit new photons and so on! How can we handle this situation? We observe that a diagram of type ”a" can be split in two parts that look like the first diagram on this page while ”b” and "c” cannot be split in this way. This means that a ”real” exchanged electron (the thick line) must be written as a sum:



where each "blob" contains a sum of all the complicated exchanges that cannot be further separated. Suppose now that we denote the contribution from the blob with *B*. The lines will be the expression for an internal line that we know and call Π :

$$\Pi = \frac{1}{m_0^2 - P^2}$$

We have here introduced the "bare" mass m_0 of the electron. The sum of diagrams can then be written

$$\Pi = \frac{1}{m_0^2 - P^2}$$

$$\Pi + \Pi B \Pi + \Pi B \Pi B \Pi + \ldots =$$

$$\Pi\left(1 + B\Pi + B\Pi B\Pi + ..\right) = \frac{\Pi}{1 - B\Pi}$$

Inserting the expression for Π we have

$$\frac{\Pi}{1 - B\Pi} = \frac{1}{m_0^2 - P^2} \cdot \frac{1}{1 - B/\left(m_0^2 - P^2\right)} = \frac{1}{m_0^2 - P^2 - B}$$

Now comes the renormalisation trick. The bare mass is not observable because when we look at an electron it is always surrounded by its virtual photons. This means that the bare mass can be set to anything convenient. So if we now define the *observable* electron mass *m* by

$$m^2 = m_0^2 - B$$

we see that the thick exchanged electron line will be represented by

$$\frac{1}{m^2 - P^2}$$

that is the same expression as before but with the bare mass changed to be the observable mass. This is called mass renormalisation. It can be done in the same way also for an exchanged photon. Due to properties of the QED it remains massless.

In the same way it turns out that we can redefine an external line to include all kinds of ”blobs". This is called renormalisation of the *wave function*.

Also the electric charge needs to be renormalised. Imagine an electron as a point charge. Very close to this charge the electric field will be extremely strong and contain a lot of energy. This means creation of virtual electron positron pairs where the positrons will be attracted toward the point charge while the electrons will be repelled. This is called vacuum polarisation and has measurable consequences for instance for the energy levels in a hydrogen atom. What you observe now will be the original point charge surrounded by a "screen" of positive charge and the *observed* electric charge will be less than the (unknown and unobservable) point charge in the centre. The renormalisation is now to take the value of the point charge such that the observable charge will be exactly what we measure experimentally $-e$, numerically $e = 1.602\cdot 10^{-19}$ C.

This also tells us that as we investigate the charge of the electron with particles of higher and higher energy, we will penetrate deeper and deeper inside the screen and the observed electric charge will change. This is actually observed experimentally.

Not all field theories that you can invent have the properties above. Not all theories are renormalisable. However it turns out that a special class of theories that have what is called local gauge symmetry are automatically renormalisable. We will investigate gauge symmetry in the next section and find that this symmetry actually performs another miracle with the theory.

Gauge symmetries

Field theories start with a Lagrangian that is a function of the fields and their derivatives. The Lagrangian must be relativistically invariant. Suppose now that we want to develop a field theory for a scalar field for charged particles $\phi(x)$ and a field $\overline{\phi}(x)$ (here the complex conjugate field) for the antiparticles. x stands for space and time coordinates. (For the moment we will use units such that the speed of light = 1.) The simplest non-trivial relativistic Lagrangian will then be

$$L\left(\phi,\overline{\phi},\partial_\mu\phi,\partial_\mu\overline{\phi}\right)=\partial_\mu\phi\partial_\mu\overline{\phi}+m^2\phi\overline{\phi}$$

We consider here the fields and their derivatives as "generalised coordinates". As usual in tensor calculus the repeated index μ means summation over this index. The Lagrangian equations are then given by

$$\frac{\partial L}{\partial\phi}-\partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\phi\right)}=0 \quad \frac{\partial L}{\partial\overline{\phi}}-\partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\overline{\phi}\right)}=0$$

The resulting equations will be

$$\Box\phi-m^2\phi=0 \quad \Box\overline{\phi}-m^2\overline{\phi}=0$$

These are nice equations because they admit solutions in the form of waves. They are called *Klein-Gordon equations*. Inserting a plane wave solution

$$\phi=Ne^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-E\cdot t)}$$

we immediately get $E^2-\mathbf{p}^2=m^2$, the correct relation between energy, momentum and mass, if we interpret the constant m in the Lagrangian as the mass of the particle.

We now add the condition that the Lagrangian should be invariant under a *global gauge transformation*:

$$\phi \to \phi \cdot e^{iq\Lambda} \quad \bar{\phi} \to \bar{\phi} \cdot e^{-iq\Lambda}$$

This is just a condition that we should always be able to multiply the field by an arbitrary phase. It is easy to see that the Lagrangian is invariant under this transformation. Actually as with all symmetries, this implies that there is a corresponding conserved quantity, in this case it can be shown that the gauge symmetry implies that the total charge is conserved. It is interesting to look at transformation from another point of view. Introduce two *real* fields and write

$$\phi = \phi_1 + i\phi_2 \quad \bar{\phi} = \phi_1 - i\phi_2$$

Then it is easy to show that the real fields will transform like

$$\phi_1 \to \phi_1 \cos q\Lambda + \phi_2 \sin q\Lambda$$

$$\phi_2 \to -\phi_1 \sin q\Lambda + \phi_2 \cos q\Lambda$$

i.e. a rotation in ϕ space. The group corresponding to such a rotation is called U(1). Another way of interpreting the gauge transformation is that we are allowed to use the label positive / negative charge as we like, as long as we are consistent.

The next step is to require that it ought to be possible to choose the phase A different indifferent space-time points as long as we change Λ continuously, i.e. we let Λ be a function of *x* (space-time). Then we will have a *local gauge transformation*:

$$\phi \to \phi \cdot e^{iq\Lambda(x)} \quad \bar{\phi} \to \bar{\phi} \cdot e^{-iq\Lambda(x)}$$

It now turns out that our Lagrangian is *not* invariant under this transformation. The only way to get it invariant is to introduce a new "photon" vector field A_μ that transforms under a local gauge transformation like

$$A_\mu \to A_\mu - \partial_\mu \Lambda$$

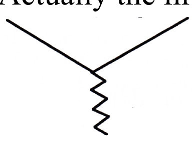
and to add some extra terms to the Lagrangian

$$L = \partial_\mu \phi \partial_\mu \bar{\phi} + m^2 \phi \bar{\phi} + J_\mu A^\mu + q^2 A_\mu A^\mu \phi \bar{\phi}$$

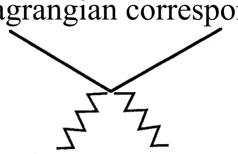
where ‘

$$J_\mu = iq\left(\bar{\phi}\, \partial_\mu \phi - \phi \partial_\mu \bar{\phi}\right)$$

This Lagrangian has local gauge symmetry. The extra terms correspond to an interaction between the "photon" = gauge particle field and the original particles. Actually the first extra term in the Lagrangian corresponds to vertices like



and the last term to



The parameter *q* corresponds to the coupling strength. We can also introduce an extra (gauge invariant) term in the Lagrangian that will generate a Klein-Gordon equation for the "photon" and we recover a kind of Maxwell's equations. However if we also try to give the "photon" a mass, which can be done by adding a term $M^2 A_\mu A^\mu$, it is impossible to get a gauge invariant Lagrangian.

Here we did the gauge transformation calculations for a scalar field in order to somewhat simplify the mathematics. However, if we repeat the game with the spinor fields of the electron and positron we get *exactly* the correct QED interaction, we don't get diagrams like the last one above. We also recover Maxwell equations.

The important points of this are:
1) With only electrons and positrons we *cannot* construct a local gauge invariant theory.
2) We need to introduce also a vector field A_μ that can be identified with the photon.
3) The particle corresponding to this vector field must be massless.
4) The gauge symmetry gives us exactly the correct form of the interaction.
5) It can be shown that the resulting theory is renormalisable.

The marvellous thing is that we can now extend these ideas to the weak and strong interactions.

We start with the weak interaction. Suppose that we want to have the freedom to label the up and down quarks as we like (or the neutrino and electron). We denote the corresponding fields by *u* and *d*. It turns out that we need a slightly more complex gauge transformation. We will have to introduce *three* vector bosons, W_1, W_2, W_3. The fundamental interaction term will look like

$$\sum_{k=1}^{3} g\cdot\begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}\sigma_k W_k \begin{pmatrix} u \\ d \end{pmatrix}$$ where we have repressed Lorentz indices.

Observe that we now have a *matrix* of fields. The quantities σ_k are (not very surprisingly) the Pauli matrices connected with the group SU(2)

$$\sigma_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If we insert these matrices we get

$$\frac{g}{2}\left[\begin{array}{l} \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_1 \begin{pmatrix} u \\ d \end{pmatrix} + \\ \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_2 \begin{pmatrix} u \\ d \end{pmatrix} + \\ \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_3 \begin{pmatrix} u \\ d \end{pmatrix} \end{array}\right]$$

or

$$\frac{g}{2}\left[\begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix} W_1 \begin{pmatrix} d \\ u \end{pmatrix} + i\begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix} W_2 \begin{pmatrix} -d \\ u \end{pmatrix} + \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix} W_3 \begin{pmatrix} u \\ -d \end{pmatrix}\right]$$

or

$$\frac{g}{2}\left[\left(\bar{u}d + \bar{d}u\right)W_1 + i\left(-\bar{u}d + \bar{d}u\right)W_2 + \left(\bar{u}u - \bar{d}d\right)W_3\right]$$

Rearranging

$$\frac{g}{\sqrt{2}}\bar{u}d\frac{1}{\sqrt{2}}\left(W_1 - iW_2\right) + \frac{g}{\sqrt{2}}\bar{d}u\frac{1}{\sqrt{2}}\left(W_1 + iW_2\right) + \frac{g}{2}\left(\bar{u}u - \bar{d}d\right)W_3$$

Finally we make the identification with the real vector bosons

$$W^+ = \frac{1}{\sqrt{2}}\left(W_1 - iW_2\right) \quad W^- = \frac{1}{\sqrt{2}}\left(W_1 + iW_2\right) \quad W^0 = W_3$$

to get

$$\frac{g}{\sqrt{2}}\bar{u}dW^{+}+\frac{g}{\sqrt{2}}\bar{d}uW^{-}+\frac{g}{2}\left(\bar{u}u-\bar{d}d\right)W^{0}$$

As you see this gives an explanation of the earlier maybe puzzling coefficients $1/\sqrt{2}$ and 1/2 in the electroweak model. Besides, there will be interaction terms in the Lagrangian that correspond to a *WWW* and a *WWWW* interaction vertex. Because of the non-linearity of the equations the coupling constant *g* must have a fixed (but unknown) value. Observe that the gauge bosons *W* must fundamentally be massless and we need the Higgs mechanism to generate their mass.

Finally we outline how the gauge symmetry gives the strong interaction. Here we want to be able to put the colour labels *R*, *G*, and *B* as we want. We will then have to introduce *eight* vector boson fields, G_k, k = 1..8, the *gluons:*

$$\sum_{k=1}^{8} g_s \begin{pmatrix} \bar{q}_R & \bar{q}_G & \bar{q}_B \end{pmatrix} T_k G_k \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}$$

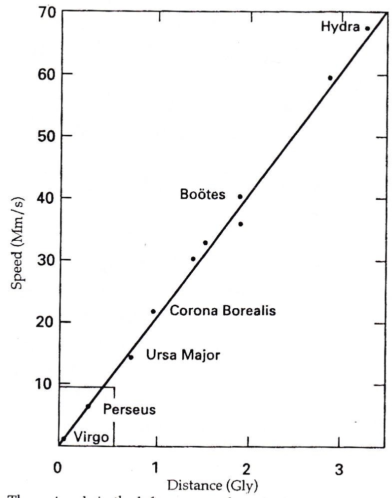
T_k are 3 x 3 SU(3) matrices and g_s is the fundamental strong coupling constant. This expression will completely describe the couplings between the quarks and the gluons. As in the weak interaction there will also be interaction terms corresponding to *GGG* and *GGGG* interactions.

Is it possible by using gauge symmetry in some higher dimension to unify all the interactions? In such gauge theories there appear a lot of new gauge bosons that we have to assume to have very large masses (by some Higgs mechanism) as they have not yet been seen experimentally. In such theories it would also be possible for leptons to transform into quarks and vice versa and the proton would be unstable. The simplest of these enlarged symmetries predict that the lifetime of the proton would be of the order 10^{31} years. Experiments have, however, shown that this lifetime is at least 10^{32} years.

Exercise: Use a simple argument to deduce that the lifetime of the proton must be quite large. You can actually very easily estimate a lower limit for it! Hint: Your body contains a lot of protons.

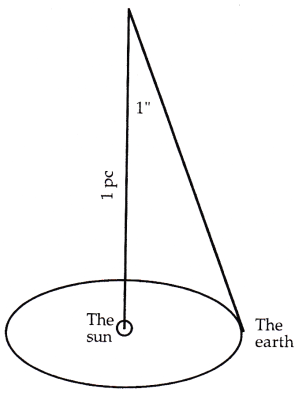
Another rather attractive theory that unifies the forces is called supersymmetry. Here you assume that each half-integer spin particle has a partner that has integer spin and vice versa. The partner of the electron is called the selectron (with the same charge and couplings), the squarks are the partners of the quarks, the sneutrinos to the neutrinos. Corresponding to the photon there is a half integer spin photino, to the W:s and Z the winos and the zino. No supersymmetric partners have so far been seen experimentally but supersymmetric particles have been suggested as candidates for the "dark mass" in the universe.
There are also the string theories that in a very elegant way include also the gravitation among the forces. Some results of this theory are quite promising; the problem is that so far the theory has not been able to give predictions that can be tested experimentally.

Chapter 7. The Big Bang



The rectangle in the left corner encloses Hubble's original data

If we study spectral lines from distant galaxies we find (in average) that these lines are displaced toward the red end of the spectrum. This can be interpreted as if the galaxies are moving away from us. Their speeds turn out to be proportional the distance from us. This is Hubble 's law and according to latest measurements, the constant of proportionality, Hubble 's parameter has a present value close to 70 km/s /Mpc. The figure to the right shows some observational data. (Note that Hubble's parameter is not a physical constant in the normal sense. It is a property of the Universe that varies with time.)



1 ly (lightyear) is equal to 0.307 pc (parsec) where 1 pc is the distance to an object that will have a parallax of 1 arc second (1 / 3600 of a degree) with the radius of the earth's orbit as a base line.

Expressed in metrical units 1 pc = $3.086 \cdot 10^{16}$ m.
1 ly = $9.460 \cdot 10^{15}$ m.
The fact that the speed is proportional to the distance can be interpreted as an expansion of the entire universe and from Hubble's law we can make a crude estimate of the start of the expansion. We have

$$v = H_0 r$$

where v is the speed, H_0 the present value of Hubble's parameter and r the distance. If we write the expression above in the form $r = v / H_0$, we see that it is formally the same as for the distance covered with a constant speed during the time interval $1/H_0$. The time $1/H_0$ will then be an estimate of the time since the expansion began.

Inserting the numerical value of Hubble's parameter we get the Hubble time $t_H = 1/H_0 \approx 4.4 \cdot 10^{17} s \approx 15 \cdot 10^9$ years. The estimate is crude because, as we will see later, neither the Hubble parameter nor the expansion speed is constant.

We can actually have a two-dimensional picture of the expansion by imagining the galaxies as patches on a rubber balloon that we inflate. It is then easy to see that two patches will move away from each other with a speed that is precisely proportional to the distance between the patches. Our discussion indicates that the expansion started with what is called the Big Bang, an event that happened about 10 000-15 000 million years ago. Note that our discussion does not say anything about the question whether the universe is finite or infinite.

We also note that the expansion will look the same from any of the patches on the balloon, something which indicates that the universe is homogeneous (the same in every point) and isotropic (the same in every direction). This assumption is called the *cosmological principle*.

We will now investigate the Hubble expansion in more detail. We will use Newtonian mechanics. This is actually not correct but will give essentially correct results since we choose equations of a form that we know look the same in general relativity.
We start by looking what happens when we throw a stone from the surface of the earth upwards in the gravitational field of the earth. As we increase the initial speed of the stone it will go higher and higher before it turns and falls back. With a sufficiently high initial velocity, the stone will never fall back.

We use the law of energy conservation

$$\frac{mv_i^2}{2} - G\frac{mM}{R} = \frac{mv^2}{2} - G\frac{mM}{r}$$

Here m is the mass of the stone, v_i its initial speed, v its final speed, r its final distance from the earth centre, M the mass of the earth, R the radius of the earth. G is the universal gravitational constant. Rearranging we find

$$v_i^2 - v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$$

The critical initial speed, the escape speed, will be reached when $r \to \infty$ and $v \to 0$ that gives

$$v_{esc} = \sqrt{2GM/R}$$

Now we use that a particle on the surface of a homogeneous massive sphere will feel a gravitational force as if the mass M inside the sphere was concentrated in the centre of the sphere. Now imagine a sphere with radius R_0 in the universe and a galaxy on the surface of this sphere. Using the same reasoning as before we can calculate the escape speed

$$v_{esc,0} = \sqrt{2GM/R_0} = R_0\sqrt{\frac{8\pi}{3}G\rho_0}$$

where we have introduced the present density ρ_0 of the universe. But by Hubble's law the speed of the galaxy at the surface of the sphere is

$$v_0 = H_0R_0$$

So if $v_0 > v_{esc}$ the universe will expand forever while if $v_0 < v_{esc}$ it will collapse in the Big Crunch. The limiting case is when

$$R_0\sqrt{\frac{8\pi}{3}G\rho_0} = H_0R_0$$

or

$$H_0 = \sqrt{\frac{8\pi}{3}G\rho_0} \text{ or } \rho_0 = \frac{3H_0^2}{8\pi G}$$

Inserting numbers we get a critical density of about 10^{-26} kg / m^3 which corresponds to about six protons per cubic meter. If the density of the universe is larger than this, the universe will collapse, if not, it will expand forever.

Our Newtonian approach gives the correct result but is logically faulty. If we had used say another sphere that was touching the original sphere at the location of the galaxy, the gravitational force would apparently have pointed in the opposite direction! However, in Einstein's theory of general relativity it is possible to derive what is called Birkhoff's theorem that says that a galaxy on the surface of a sphere in a homogeneous universe will be influenced only by the mass inside the sphere, irrespectively of how the sphere is chosen. Which means that we can neglect everything in the rest of the universe that is outside this sphere.

We derived our results above for the present universe. But it must of course be valid for any time *t*. This means we have

$$\rho_{critical}(t) = \frac{3H^2(t)}{8\pi G}$$

We now define the density parameter

$$\Omega(t) = \frac{\rho(t)}{\rho_{critical}(t)} = \frac{8\pi G\rho(t)}{3H^2(t)}$$

A density parameter that is less than or equal to 1 then means eternal expansion, a density parameter larger than 1 means collapse.

Using the measured value of the present Hubble parameter and estimating the present mass density in the universe, we can calculate the density parameter. It then turns out that the visible matter in the galaxies will only give Ω = 0.01. As we will see later on we know that there is quite a lot of invisible ordinary matter in the galaxies. This invisible normal matter can be estimated to increase Ω to about 0.05. Studies of the rotational speeds of stars in galaxies indicate that there is still more so called dark matter (see chapter9). Still the density parameter will only be of the order of 0.3. As we will soon see, there are reasons to believe that the density parameter is precisely equal to 1. This means that either there must be about 3 times as much mass or energy again in the form of "cosmic dark matter", invisible and of unknown composition. Another way to save the situation is to introduce the cosmological constant, more about that in the last section of this chapter.

Details of the Big Bang

From the earlier discussion the can write the equation for the radial speed of a galaxy on the surface of a sphere with radius *R* as

$$v^2 - \frac{2GM}{R} = \text{constant}$$

The constant obviously has the dimension of velocity squared and we write

$$v^2 - \frac{GM}{R} = k{\cdot}c^2$$

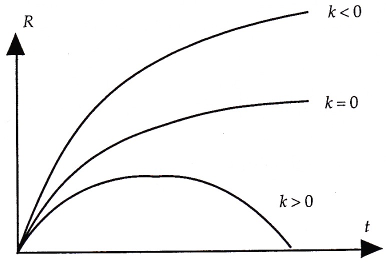
where *c* is the speed of light and *k* is a dimensionless constant. From our earlier results we then have

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi GR^2\rho(t)}{3} = \left(H(t)R(t)\right)^2\left[1-\Omega(t)\right] = -k{\cdot}c^2$$

We note that for a critical density parameter Ω =1, we obviously have $k = 0$ which in turn means that the total energy at the surface of the sphere is zero. But this is valid for any size of the sphere which implies that the total energy of the universe is zero for critical density. Rewriting the equation we have

$$\frac{dR}{dt} = \pm\sqrt{\frac{2GM}{R} - kc^2}$$

We know empirically that our universe expands and choose the positive root. If the value of *k* is negative, the expression inside the root sign is always positive, the expansion speed is then always positive and R will increase forever although by time in a slower pace. If $k = 0$ the expansion speed will go to zero as R goes to infinity. But if $k < 0$ there will be a finite value of R where the expansion speed is zero, we then have to choose the negative sign of the root to study the further development. Graphically we can picture the development with time like this



In general relativity a universe with $k < 0$ is open (in two space dimensions it would correspond to a saddle surface), and a universe with $k > 0$ is closed (corresponding to a sphere in two dimensions). The special case where $k = 0$ is called a flat universe, (in two dimensions corresponding to a plane).

Note that in the beginning all the curves are very close to each other. If we assume that the present universe is not too far from the origin of the Big Bang, we can consider only the situation $k = 0$ and have

$$\frac{dR}{dt} = \sqrt{\frac{2GM}{R}} \quad \text{or} \quad R^{1/2}\frac{dR}{dt} = \sqrt{2GM}$$

or $R = \left(\frac{3}{2}\sqrt{2GM}\right)^{2/3} t^{2/3} \propto t^{2/3}$ where we have assumed $R(t = 0) = 0$.

If we now use that $M = 4\pi R^3\rho/3$ we get after some rearrangement

$$t = \frac{2}{3}\sqrt{\frac{3}{8\pi G}}\rho^{-1/2}$$

This is a "cosmic clock", by measuring the density of the universe we can calculate the age of the universe. The present age of the universe can be calculated by inserting the present density

$$t_0 = \frac{2}{3}\sqrt{\frac{3}{8\pi G}}\rho_0^{-1/2} = \frac{2}{3H_0}\sqrt{\frac{3H_0^2}{8\pi G\rho_0}} = \frac{2}{3H_0}\sqrt{\frac{1}{\Omega_0}}$$

Assuming that our universe is critical and inserting numerical values we get

$$t_0 = \frac{2}{3}t_H \approx 10^{10}\,\text{years.}$$

This is actually a little too small as the astronomers have found stars that seem to be older than this. We will return to this problem later on.

Here we also note that from

$$\left(H(t)R(t)\right)^2\left[1 - \Omega(t)\right] = -k\cdot c^2 = \text{constan}$$

we have

$$\left(H(t)R(t)\right)^2\left[1 - \Omega(t)\right] = \left(H_0R_0\right)^2\left[1 - \Omega_0\right]$$

But as $R \propto t^{2/3}$ we must have $v = \dfrac{dR}{dt} \propto t^{-1/3}$ that gives

$$\left|\Omega(t)-1\right| = \left|\Omega_0 - 1\right|\left(\frac{t}{t_0}\right)^{2/3}$$

Now assume that today Ω_0 = 0.1. Today we have $t_0 = 3\cdot 10^{17}\,\mathrm{s}$. Let us calculate the density parameter 1 s after the Big Bang. We then have

$$\left|\Omega(t)-1\right| = \left|0.1-1\right|\left(\frac{1}{3\cdot 10^{17}\,\mathrm{s}}\right)^{2/3} \approx 10^{-11}$$

At that time the density parameter must have been *extremely* close to 1!! It is very difficult to understand how Nature could make the density parameter that close (but not equal to) 1. A natural assumption is to say that it is exactly equal to 1. In that case, the constant $k = 0$ and the density parameter will always be equal to 1. It is then a natural question to ask how the density parameter became exactly equal to 1. We will see how to solve that problem in a later chapter.

Radiation in the universe

Our derivation above is not complete. Electromagnetic radiation contains energy and thus by Einstein contributes to the mass inside the sphere. The equivalent mass of the radiation is given by

$$M_{rad} = E_{rad} / c^2$$

We therefore suspect, and this is confirmed by the theory of general relativity, that the mass of the sphere must be redefined as

$$M(R) = M_{matter}(R) + M_{rad}$$

As the sphere expands the amount of matter doesn't change but this is not true for the mass coming from radiation.

Our first interpretation of the displacement of the spectral lines of distant galaxies was that they actually moved through space with a certain speed. This, however, is not correct. The correct interpretation is that we see the galaxies recede from us because space itself expands. We don't observe this inside the galaxies, because gravitation will keep matter together. If we compare this with our expanding balloon, we could say that the galaxies are spots of glue on the surface of the balloon and when we inflate the balloon the area between the spots expands while the spots keep the same size. Now imagine a light wave drawn as a wavy line on the balloon. As the balloon inflates the wavelength will obviously increase, the light will be more red. This is the correct interpretation of the "Doppler" shift of the spectra. This immediately tells us that the wavelength λ of the light
is proportional to R. So

$$R \propto \lambda = c / f = \frac{hc}{hf} \propto 1/E \quad \text{or} \quad E \propto 1/R$$

This means that when the universe was small, in its early evolution, the contribution to its mass was radiation dominated. We call this the *radiation dominated era*. We can redo our calculations for this case. We have

$$\frac{dR}{dt} = \sqrt{\frac{2GM}{R} - kc^2} = \sqrt{\frac{2GE}{c^2 R} - kc^2} = \sqrt{\frac{2GK}{c^2 R^2} - kc^2}$$

where K is some proportionality constant. Assuming as before that $k = 0$ we get

$$\frac{dR}{dt} = \sqrt{\frac{2GK}{c^2 R^2}} \quad \text{or} \quad R\frac{dR}{dt} = \sqrt{\frac{2GK}{c^2}} \quad \text{or} \quad R = \left(8GK / c^2\right)^{1/4} t^{1/2} \quad \text{or} \quad R \propto t^{1/2}$$

For the cosmic clock we get

$$t = \frac{1}{2}\frac{R^2}{\sqrt{2GK / c^2}} = \frac{1}{2}\frac{R^{3/2}}{\sqrt{2GK / Rc^2}} = \frac{1}{2}\frac{R^{3/2}}{\sqrt{2GM_{rad}}} = \frac{1}{2}\sqrt{\frac{3}{8\pi G}}\rho^{-1/2}$$

The density means the equivalent radiation mass per volume. We see that the time still has the same functional dependence on the density as before but that the "size" of the universe now is proportional to the square root of time.

The cosmic background radiation

In the beginning the radiation was in thermal equilibrium with the matter: the photons interacted with free charged electric particles like protons and electrons. As long as this happened the average energy of the photons was equal to the average particle energy. But as the universe expanded, the energy of the photons and particles decreased and at some time the electrons and protons combined to form neutral atoms.

At that time the radiation more or less stopped interacting with matter, the interaction cross-section for photons and atoms is very small (see chapter 3 on the Compton scattering). We can actually estimate when this happened. Typically ionisation energies are of the order of 1 eV. Using a somewhat more detailed calculation it can be shown that neutral atoms were created when the temperature of the radiation (and the particles) were about 4000 K. The electromagnetic radiation in the universe is blackbody radiation and we have (by Planck's law) a precise relation between the (mass equivalent) energy density of the radiation and its temperature

$$\rho c^2 = \frac{4}{c}\sigma_{SB} T^4$$

where σ_{SB} is the constant in Stefan-Boltzmann's law. Inserting this in our equation above gives us

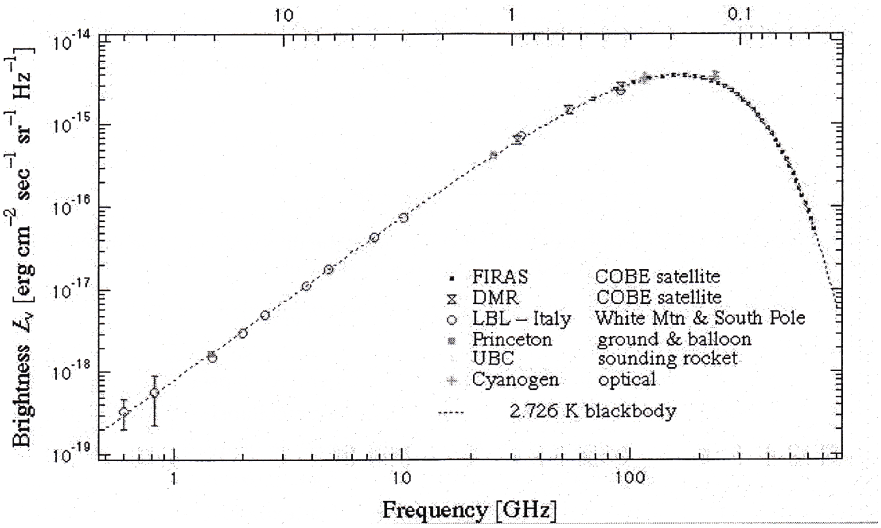
$$t = \frac{1}{2}\sqrt{\frac{3c^2}{32\pi G \sigma_{SB}}} T^{-2} \propto T^{-2}$$

If we use the value T = 4000 K we find a time t_{free} = 500 000 years. In this computation we have assumed that the universe was radiation dominated up to this point which we will justify in a moment. After this moment, the radiation will stop interacting with matter in the universe and will lose energy only due to the expansion, we have $E_{rad} \propto R^{-1}$ and $E_{rad} \propto T_{rad}$, thus $T_{rad} \propto R^{-1}$. (Matter and radiation will now not necessarily have the same temperature as they are not in thermal contact.) We can then calculate the present temperature of the radiation

$$T_0 = T_{free}\frac{R_{free}}{R_0} = T_{free}\left(\frac{t_{free}}{t_0}\right)^{2/3} = 4000\left(\frac{5\cdot 10^5}{10^{10}}\right)^{2/3} \approx 5\,\text{K}$$

This is remarkably close to the measured value 2.726 K of the temperature of the cosmic background radiation, first discovered by Arno Penzias and Robert Wilson in 1964. The equivalent mass density of the radiation today is about $4.5\cdot 10^{-31}$ kg / m^3, about 1/1000 of the matter density. At the time where the radiation stopped interacting with matter the equivalent radiation mass density was $2\cdot 10^{-18}$ kg / m^3 while the matter density was about $4\cdot 10^{-18}$ kg / m^3, almost the same. By a strange coincidence the time when the radiation got free (the universe became transparent for light) also the universe changed from being radiation dominated to be *matter dominated.*

The spectrum of the background radiation closely conforms with a Planck law blackbody radiation of 2.726 K. See the attached figure with measured data.



The cosmological constant

We remarked earlier that the present age of the universe was uncomfortably close to the age of the oldest stars. Also we had problems with that the amount of matter was not enough to make the universe critical. One way of curing our model of the universe is to introduce a constant vacuum matter (energy$\cdot c^2$) density Λ. If we then include all kinds of mass/energy in our original equation of motion we have

$$v^2 - \frac{2G}{R}\left(\tfrac{4}{3}\pi\rho_{matter}R^3 + \tfrac{4}{3}\pi\rho_{rad}R^3 + \tfrac{4}{3}\pi\rho_{vac}R^3\right) = -kc^2$$

We know that the radiation density is proportional to $T^4 \propto R^{-4}$ and is important only in the radiation dominated era. If the vacuum energy density is constant the corresponding term in the expression above can be neglected in the early development of the universe but will be dominant in the later phase of the development. If we look at our present universe and neglect the radiation term we have

$$v^2 - \frac{2G}{R}\left(\tfrac{4}{3}\pi\rho_{matter}R^3 + \tfrac{4}{3}\pi\rho_{vac}R^3\right) = -kc^2 = \left(HR\right)^2\left(1 - \Omega_{matter} - \frac{\Lambda}{3H^2}\right)$$

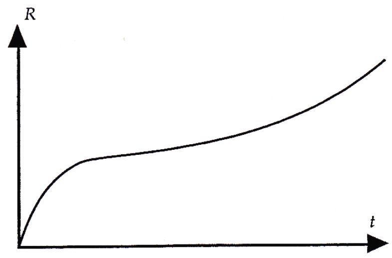
where $\Lambda = 8\pi G\rho_{vac}$ is the so called *cosmological constant*. You will then see that we have an "effective" density parameter of

$$\Omega_{effective} = \Omega_{matter} + \frac{\Lambda}{3H^2} = \Omega_{matter} + \Omega_\Lambda$$

This means that we can have a flat space ($\Omega_{effetive}=1$) without having $\Omega_{matter}=1$

We also then have

$$\frac{dR}{dt}=\sqrt{\frac{2GM}{R}+\tfrac{1}{3}\Lambda R^2-kc^2}$$



We can now have a multitude of different possible universa. For instance if we choose $k = 0$, we see that for small R, the universe will behave as the earlier critical universe but for large R the second term in the root will dominate and give an expanding universe. The "size" of the universe could then develop as the sketch shows

If we then live in the later phase of the development, the universe could be much older than t_H. Some of the results are summarized in the table below

Ω_{matter}	Ω_Λ	$\Omega_{effective}$	Present age of the universe	Type of universe
1	-1	0	8Gy	Open, recollapses
1.5	-0.5	1	8Gy	Flat, recollapses
2	-0.5	1.5	8Gy	Closed, recollapses
0.3	0	0.3	12Gy	Open, expands
1	0	1	10Gy	Flat, expands
2	0	2	8Gy	Closed, recollapses
0.3	0.3	0.6	12Gy	Open, expands
0.3	0.7	1	12Gy	Flat, expands
0.3	1.7	2	40Gy	Closed, loitering
0.3	2	2.3	∞	Closed, bouncing

In 1999 observational situation can be summarized as

Technique	Value of Ω_Λ	Limit
Gravitational lensing Kochanek(1966)	-	<0.66
Gravitational lensing Myungshin(1997)	0.64	–
Gravitational lensing Chiba & Yoshii(1997)	≈ 0.8	-
Gravitational lensing Chiba & Yoshii(1999)	0.7	-
Supernovae redshift Perlmutter(1997)	0.06	<0.51
Supernovae redshift Riess(1998)	0.68	>0
Supernovae redshift Riess(1998)	0.84	>0
Supernovae redshift Perlmutter(1999)	0.71	>0

The most recent observations indicate that we live in a flat universe with $\Omega_{matter} = 0.1$ and $\Omega_{matter} = 0.9$.

There are theoretical arguments for the existence of a cosmological constant. According to quantum mechanics there should be a zero point energy due to the creation of virtual particle-antiparticle pairs in the vacuum. The concept of this vacuum energy has was been experimentally confirmed in1996 through the Casimir effect where two uncharged conducting plates attract each other due to vacuum fluctuations.

We can estimate the vacuum energy by considering spinless particles that can be represented by a collection of harmonic oscillators. The zero point energy of a harmonic oscillator is given by $\hbar\omega / 2$.
If we sum all the possible frequencies we get

$$E_0 = \sum_{\omega} \frac{\hbar\omega}{2}$$

We can convert the sum to an integral over the density of states in a volume V.

$$E_0 = \frac{V}{(2\pi)^3} \int \frac{\hbar\omega}{2} d^3k = \frac{V}{(2\pi)^3} 4\pi \int_0^{\omega_{\max}} \frac{\hbar\omega}{2c^3} \omega^2 d\omega = \frac{V(\hbar\omega_{\max})^4}{16\pi^2 c^3 \hbar^3}$$

The upper limit of the integration can be taken as the Planck energy, $1.2 \cdot 10^{19}$GeV, because we guess that our ordinary physics should be more or less valid up to this point. This gives us an extra equivalent mass density of

$$\rho_{vac} = \frac{E_0}{V} \cdot \frac{1}{c^2} = \frac{E_{Planck}^4}{16\pi^2 c^5 \hbar^3}$$

This would result in

$$\Omega_{\Lambda} = \frac{8\pi G \rho_{vac}}{3H_0^2} \approx 10^{120}$$

This is in extreme (!!) conflict with the observed values above. It could be that if we - include the contribution of all the particles in the standard model that by some miracle the contributions would more or less cancel to give the observed value.

Review:
Give some observational facts that support the Big Bang idea. Define the critical density. Define the density parameter. Derive the functional dependence of the size of the universe as a function of time in a critical universe (both matter and radiation dominated). Derive the "cosmic clock”. Show that the density parameter must have been very close to 1 close to the Big Bang. Show that our theory predicts a present cosmic background radiation temperature of some Kelvins. Why is a cosmological constant an interesting addition to the Big Bang theory?

Problems:
1. What is the time elapsed from the Big Bang until the tauons annihilated? ($2 \cdot 10^{-7}$ s)
Hint: After 500 000 years when the radiation was released, the temperature of the universe was 4000 K.

2. Start from the following facts:
The photon density (number density) in our present universe is 10^8 m^{-3}.
The nucleon density in our present universe is 1 m^{-3}. The Hubble parameter is 75 km/s /Mpc. The neutrinos in the universe were released when the universe became "transparent" to them i.e. when their mean free path was of the same order of magnitude as the size of the universe. This happened some seconds after Big Bang when the temperature of the universe was about $5 \cdot 10^9$ K. Before the release, the neutrinos were in thermal equilibrium with other particles. After the release the temperature of the neutrinos decrease as the size of the universe increases, independent of the other kinds of particles. Assume that the present value of the density parameter $\Omega = 1$ i. e has the critical value. Furthermore, assume that the "missing mass" can be explained by that the neutrinos have a mass. What approximate value does this model give for the sum of the masses of the three kinds of neutrinos: $(m_{\nu e} + m_{\nu\mu} + m_{\nu\tau}) \cdot c^2$? Give the answer using the unit electron volts. (10 – 100 eV) Hint: First show that the present number density of neutrinos is of the order of 10^8m^{-3}, the same as the present number density of photons.

3. As the universe had cooled such that the electrons and atomic nuclei had formed neutral atoms the universe became transparent. Light with different wave-lengths: radio waves, visible light, X-rays and photons with still higher energies (maximal observed energy is 70 TeV) can be used to study objects as far away as several billions of light-years. As the universe contains a lot of photons, this would not have been possible had the cross-section of $\gamma\gamma \to \gamma\gamma$ not been very small for normal (small) photon energies. At very high energies other reactions like $\gamma\gamma \to X$ are possible.
a) What final state should be easiest to produce in the latter reaction? (e^+e^-)
b) Draw a Feynman diagrams for this process. (If you don't know the answer of part a), draw a diagram for some other possible final state and use it for the following questions)
c) Study this reaction in the system in the COM frame of the photons is. Determine the smallest possible energy E_0 for the reaction to be possible? ($E_0 = 2m_e c^2$)
d) Estimate the order of magnitude of the cross section 0' of the process. Assume that the energy is of the same order of magnitude as the minimal energy E_0. Use dimensional analysis and that the dimensionless parameter $\alpha = e^2 / \left(4\pi\varepsilon_0 \hbar c\right)$ =1/137.(10^{-30} m^2)
e) What is the energy of the photons in the cosmic background radiation? What energy does a high-energy photon need to interact with the background radiation in the reaction above? (10^{15} eV)
f) What is the mean free path for high-energy photons expressed in the cross section σ and number density n_γ ny of background photons per volume? What is the approximate value of n_γ? Use this and the cross section estimated above to discuss how this could influence the observation of very distant galaxies. (10^{22} m)
g) There are also a great number of low-energy neutrinos in the universe. Draw a Feynman diagram for a possible reaction process between a neutrino and a high-energy photon. Can this reaction be neglected compared with the $\gamma\gamma$ reaction above and if so, why? ($\sigma = 10^{-11} \sigma_{\gamma\gamma}$) Hint: What would be the minimum necessary energy for this process in the C.O.M system? Use this and dimensional analysis.

Chapter 8. The evolution of the universe

We can compute the number density of photons in the universe using the Planck distribution. It is given by

$$n_\gamma = 2.404\left(\frac{kT}{\pi c\lambda\!\!\!^{-}}\right)^3$$

Inserting the temperature of the present universe we get the present number $n_{\gamma,0} = 1.3 \cdot 10^8\,\mathrm{m}^{-3}$. We can compare this with the number density of matter, about one nucleon per cubic meter. The ratio between the number of photons and nucleons is thus about 10^8. However, if we look at the distribution of energy, the present photons have typically an energy of $kT = 10^{-4}$ eV, while a typical nucleon mass corresponds to 1 GeV. This means that the energy ratio between photons and nucleons is

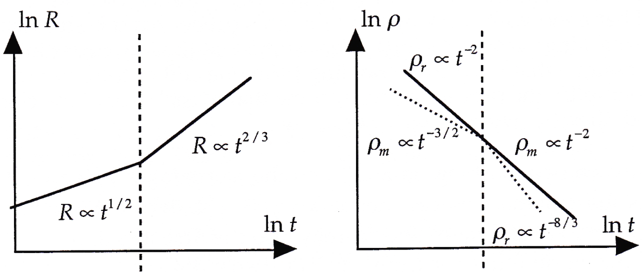
$$\frac{10^8 \cdot 10^{-4}}{1 \cdot 10^9} = 10^{-5}$$

We live now in the *matter dominated era*.

We know that in a matter dominated universe we have $R \propto t^{2/3}$. Further the matter mass density is proportional to R^{-3}, i.e. $\rho_m \propto t^{-3}$. The radiation equivalent mass density is proportional to T^4, and as $T \propto R^{-1}$ we have $\rho_r \propto R^{-4} \propto t^{-8/3}$.

In the radiation dominated universe the change is that $R \propto t^{1/2}$. This gives for the matter mass density $\rho_m \propto t^{-3/2}$, while the radiation equivalent mass density becomes $\rho_r \propto R^{-4} \propto t^{-2}$.

We illustrate these relations with the following diagrams



The dotted vertical line marks the borderline between the radiation dominated and matter dominated universe at about t = 500 000 years.

Travelling back to the past

We start today and go back in time toward the Big Bang itself.

Today
t=1.2·10^{10} years = 4·10^{17}s T = 2.726 K $\rho_m = 10^{-27}$ kg/m^3 ρ_r =10^{-32} kg/m^3

Room temperature universe
t=1.2·10^7 years = 4·10^{14}s T = 300 K $\rho_m = 10^{-21}$ kg/m^3 ρ_r =10^{-24} kg/m^3

The borderline to the radiation dominated universe
t=5·10^5 years = 10^{13}s T = 4000 K $\rho_m = 10^{-17}$ kg/m^3 ρ_r =10^{-17} kg/m^3

The leptonic era, we now produce lots of electrons, muons, and neutrinos
t=1 s T = 10^{10} K E=1 MeV $\rho_m = 10^2$ ρ_r =10^8 kg/m^3

The quark era, we now have free quarks
t=10^{-5} s T = 10^{12} K E=300 MeV

W, Z in thermal equilibrium with other particles
t=10^{-10} s T = 10^{15} K E=100 GeV

Electromagnetic, weak and strong interactions unite(?)
t=10^{-36} s T = 10^{28} K E=10^{15} GeV

The Planck time, we need a quantum theory for gravitation to continue.
t=10^{-43} s T = 10^{32} K E=10^{19} GeV $\rho = 10^{96}$ kg/m^3

Some problems with the simple Big Bang: the horizon problem and the flatness problem

We will first address the horizon problem. The problem is that the visible universe grows faster than the expansion of the universe. As time goes by, we will be able to see a larger and larger portion of the universe.

To understand this problem we consider a one-dimensional "rubber" universe. We assume that we have a rubber band with markings and that this rubber band is stretched. The distance between two markings at time t is $R(t)$. We assume that time increases in discrete steps. We further assume that a worm is crawling from one end of the band to the other with speed c. To simplify the procedure, we assume that at each tick of the clock the rubber band is stretched, the worm takes a step $c\cdot\Delta t$.

At time Δt the distance between two markings is $R(\Delta t)$ and the worm takes a step $c\cdot\Delta t$. At the next tick of the clock the distance between the markings is $R(2\Delta t)$, i.e. the rubber band has elongated by a factor $R(2\Delta t)/R(\Delta t)$ and thereby also the first step of the worm. After a new step of the worm it has consequently travelled a distance

$$s(2\Delta t) = \frac{R(2\Delta t)}{R(\Delta t)} c \cdot \Delta t + c \cdot \Delta t$$

After the next step the worm has travelled

$$s(3\Delta t) = \frac{R(3\Delta t)}{R(2\Delta t)} \left(\frac{R(2\Delta t)}{R(\Delta t)} c \cdot \Delta t + c \cdot \Delta t \right) + c \cdot \Delta t =$$

$$R(3\Delta t) c \cdot \Delta t \left(\frac{1}{R(\Delta t)} + \frac{1}{R(2\Delta t)} + \frac{1}{R(3\Delta t)} \right)$$

After N steps we then have

$$s(N\Delta t) = R(N\Delta t) \sum_{n=1}^{N} \frac{c\Delta t}{R(n\Delta t)} \xrightarrow{\Delta t \to 0} R(t) \int_0^t \frac{cdt'}{R(t')}$$

For a radiation dominated universe we have $R(t) = k \cdot t^{1/2}$ that gives

$$s(t) = ct^{1/2} \int_0^t \frac{dt'}{t^{1/2}} = 2ct$$

For a matter dominated universe with $R(t) = k' \cdot t^{2/3}$ we get $s(t) = 3ct$.

In both cases the distance that a light ray travels is proportional to t. But the size of the universe only grows as $t^{1/2}$ or $t^{2/3}$. The visible universe grows faster than the expansion. All the time we will see new parts of the universe that have never been in contact with ”our” universe. But still we observe that the cosmic background radiation is extremely isotropic and has the same temperature in every direction and from any part of the universe. This is highly astonishing, like discovering a planet that has never had any contact with the earth but looks identical!

Our second problem with the simple Big Bang is the flatness of the universe. How can the density parameter be so closely tuned to be almost exactly 1?

It turns out that both these problems can be solved by the idea of inflation.

Inflation

Let us for a moment go back to our equation of motion for the universe in chapter 7:

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G\rho(t)}{3}} R(t)$$

Suppose now that for some reason the energy density is constant. We then have the equation

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G\rho_{const}}{3}} R(t)$$

that has exponentially an growing solution:

$$R(t) = R_i e^{\sqrt{\frac{8\pi G\rho_{const}}{3}}(t-t_i)}$$

If the constant density is large enough, the universe will be ”inflated" tremendously in a very short time. This solves the horizon problem: a small part of the universe is inflated to be ”our” part of the universe, the parts outside were before the inflation in contact with us and it is not strange that when they again become part of the visible universe they are in thermal equilibrium with "us”. The inflation also solves the flatness problem: whatever the curvature the original universe had, it will look flat when inflated say by a factor of 10^{30}.

What could cause the energy density to become constant? It turns out that a Higgs medium for the Grand Unified Theory of elementary particles can deliver the necessary energy when the medium exhibits a spontaneous breakdown of symmetry. This is very much like what happens when water freezes, there is a phase transition, and as the water freezes it delivers melting heat.

In the "standard" Big Bang the inflation happened at about $t = 10^{-32}$ s and lasted about 10^{-31} s.

Nucleon and primary nuclear synthesis

When quarks and antiquarks combined into nucleons in the nucleon synthesis there must have been a slight excess of quarks as all the matter we see today is built from quarks. We can compute the necessary excess as there are three quarks in a nucleon. If the number of nucleons today is N_0 we then have

$$N_0 = \tfrac{1}{3}\left(N_q - N_{\bar{q}}\right)$$

On the other hand at the time of the nucleon synthesis, the number of quarks and anti-quarks must have been about the same as the number of photons that is the same as the number of photons today

$$N_{0,\gamma} \approx N_q \approx N_{\bar{q}}$$

Question: Why is the number of photons then the same as today? Hint: How does the photon number density depend on *R*?
This means that

$$\frac{N_q - N_{\bar{q}}}{N_q + N_{\bar{q}}} = \frac{3N_0}{2N_{0,\gamma}} \approx 10^{-8}$$

In some weak interaction reactions that we study today there are examples that show that there is a slight asymmetry between the reaction probabilities of particles and anti-particles. Similar asymmetries in Grand Unifies Theories combined with inflation would allow a slight excess of matter in an originally equal mixture of particles and anti-particles.

At the end of the leptonic era (t = 1 s), when the typical energy is about 1 MeV ($T = 10^{10}$ K) it began to be important that the neutron is slightly heavier than the proton. Up till now neutrons and protons had been freely converted into each other by collisions with electrons, positrons and neutrinos. But now it became a little more difficult to produce neutrons than protons. We can estimate the ratio of the numbers of neutrons and protons by using the Boltzmann factor

$$r = \frac{N_n}{N_p} = \frac{e^{-m_n c^2/kT}}{e^{-m_p c^2/kT}} = e^{-\left(m_n - m_p\right)c^2/kT}$$

Inserting numbers we find $r \approx 0.27$. This number has to be adjusted for the fact that the neutron has a finite lifetime (about 10 minutes) and taking this into account will give $r \approx 0.14$.

Precisely at this time the energy has decreased such that neutrons and protons can start to form deuterium nuclei. The binding energy of a deuterium nucleus is 2.2 MeV, and earlier the energy was high enough to tear the nucleons apart as soon as they united. Once the deuteron nuclei have formed they can rapidly capture more neutrons and protons or directly unite two and two to helium. At $t = 200$ s, all existing neutrons would be locked up in helium-4 nuclei, the remaining nucleons are all protons. We can now easily calculate the fraction of nucleons ending up in a helium-4 nucleus. We have

$$N_n = 2N_{He} \quad N_p = 2N_{He} + N_H$$

This gives $N_{He} = N_n / 2 = 0.07N_p \quad N_H = N_p - 2N_{He} = 0.86N_p$

and the fraction $\dfrac{4N_{He}}{N_p + N_n} = \dfrac{4 \cdot 0.07}{1.14} = 25\%$

It is a very strong support for the Big Bang theory that this is the value we observe in the universe.

Galaxy formation

At about $t = 500\,000$ years the helium and hydrogen nuclei were able to form electrically neutral atoms and the universe became transparent to the photons. Small random matter condensations started to contract under the influence of gravitation and to attract nearby matter. It can be shown that during the radiation dominated era only condensations with masses larger than the *Jeans mass* will contract within reasonably short time. The Jeans mass during this era is given by

$$M_J = \left(\frac{1}{36\pi}\right)^{1/2} \frac{c^3}{G^{3/2}\rho_{rad}^{1/2}}$$

During the matter dominated era, the Jeans mass is given by

$$M_J = \left(\frac{3}{4\pi}\right)^{1/2} \left(\frac{kT}{m_p G}\right)^{3/2} \frac{1}{\rho_{matter}^{1/2}}$$

On the other hand the photons will during the radiation dominated era try to disperse these mass concentrations and only masses larger than the so-called *Silk mass* will survive into the matter dominated era when the scattering effect of the photons disappear. The Silk mass is given by

$$M_S = \frac{4\pi}{3\rho_{matter}^{1/2}} \left(\frac{m_p ct}{\sigma}\right)^{3/2}$$

where σ is the Compton cross section that we treated in chapter 3.

During the radiation dominated era the Jeans mass increased to 10 million times the size of a typical galaxy. No condensations of the size of a galaxy could thus form during this era. Entering the matter dominated era the Jeans mass suddenly decreased to be very small and then decreased with time. But interactions with photons during the radiation dominated era would destroy condensations with a mass less than the Silk mass. Thus only condensations with a mass larger than the Silk mass would survive into the matter dominated area. The Silk mass has a numerical value of about 10^{14} solar masses not too remote from the mass of a typical galaxy, 10^{11} solar masses.

We should be able to see the fluctuations in the mass density reflected as fluctuations in the cosmic background radiation. This is exactly what has been seen recently from the COBE satellite.

Star formation

The primary galaxy cloud will further breakup in smaller condensations that eventually will form stars. As these condensations contract, they gain gravitational potential energy. You would expect that this would lead to an increase in the temperature of the gas cloud. However, the gas cloud that at this time consists of essentially hydrogen molecules has a very effective mechanism to get rid if the increased energy: the energy is stored as rotational energy and is then emitted as radiation with a wavelength of $2.8 \cdot 10^{-5}$ m, for which the gas cloud is more or less transparent. This means that the contraction will be more or less a free fall. However, some of the gravitational energy will be used to split the hydrogen molecules into hydrogen atoms at when the radius of the future star is of the order of 1000 solar radii this leads to that the gas will no longer be transparent to the emitted radiation and starts to heat up. The gas cloud has become a proto-star.

As the temperature of the protostar increases to the order of 10‘ K, corresponding to a mean energy of some electron volts, the atoms will be ionized. As the temperature is further increased different kinds of fusion reactions will start like

$$p + p \to d + e^{+} + \nu_{e}$$

$$p + d \to {}^{3}He + \gamma_{e}$$

$${}^{3}He + {}^{3}He \to {}^{4}He + p + p$$

The proto-star is now a star. This will happen at a temperature of about 10^{7} K. The fusion will develop a lot of energy and the contraction will stop. The star is now in a stable equilibrium, if it contracts the temperature in the centre will increase, boosting the fusion that increases the pressure making the star expand again. The star is a self-regulating fusion reactor. The lifetime of such a star will be inversely proportional to its mass squared, heavy stars will have a short life. Also the luminosity of the star will be proportional to its mass cubed.

Star death

For a star of the size of the sun its hydrogen fuel will last for about 10 Ga (a = annum = year). The centre of the star will now consist of a nucleus of helium surrounded by a shell of burning hydrogen. During this phase, the star inflates and becomes a *red giant*. In the central part that contracts by gravitation, the temperature increases and helium nuclei start to combine to ${}^{12}C$ nuclei in a two-step reaction. The carbon nuclei can react again with helium nuclei and form ${}^{16}O$ that can further react to form ${}^{20}Ne$. After about 60 Ma the central part of the star will consist mainly of carbon and oxygen. The outer parts have been blown out as a *planetary nebula*.

As the helium fuel is finished, the star will contract and quantum mechanical effects will start to be important. The electrons in the star form a degenerate Fermi gas.
The gravitational energy of a homogeneous sphere (the star) with radius R is inversely proportional to this radius and negative

The quantum mechanical energy of the electrons is non-relativistically inversely proportional to the square of the radius (Compare the quantum mechanical square well) and can more precisely be shown to be

$$E_g = -\frac{2GM^2}{5R} = -\frac{A}{R}$$

In the second formula we have assumed that there is one electron per hydrogen atom in the star, i. e. the number of electrons N_e is given by the ratio of the mass of the star M and the mass of the proton / neutron, m_p.

$$E_e = \left(\frac{9\pi}{4}\right)^{2/3} N_e^{5/3} \frac{\hbar^2}{2m_e R^2} = \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{m_p}\right)^{5/3} \frac{\hbar^2}{2m_e R^2} = \frac{B}{R^2}$$

Exercise: Show with a diagram sketch how the total energy depends on the radius of the star. Show that there is a minimum of the energy. Determine the equilibrium radius, and then show that it is proportional to $M^{-1/3}$.

The equilibrium radius of a star with the mass of the sun will be about 10 000 km, comparable to the radius of the earth. The star has become a *white dwarf*, a compact sphere of gas that behaves somewhat as a metal, i.e. has a large heat conductivity. Note that the gas of atomic nuclei, that have a mass that is about 2000 times the electron mass, is not degenerated. The white dwarf is essentially transparent to radiation, the degenerate electron gas cannot absorb radiation. Only a thin outer skin on the star is non-transparent and radiates thermal energy. The white dwarf will slowly cool, typical times for this cooling is 1 Ga.

If a proto-star has too small mass it will contract and reach the degenerate equilibrium radius before the temperature gets high enough for the fusion to start. Such ”stars" are called *brown dwarves*.

If the mass of the star is too high the highest electron energy levels will have relativistic energies. In this case we have

$$E_e = \frac{B'}{R}$$

i.e. this energy will only be inversely proportional to the radius. If the factor *B'* is less than the factor *A*, the star electron gas will collapse. The limiting case *A* = *B'* happens when the mass of the star is

$$M_{Ch} = \frac{1}{6\pi^{1/2} m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2}$$

the *Chandrasekhar mass*.

Inserting numerical values in the formula gives $M_{Ch} = 1.4 M_{sun}$.

As the outer parts of the star are dispersed during the fusion, even stars with considerably larger mass than the Chandrasekhar mass will end up as white dwarves.

If the star has considerably larger mass than the sun, the end will be more dramatic. As the helium fuel is used up, carbon nuclei can form magnesium and neon nuclei, a process that takes of the order of 100 years. The star contracts and neon and oxygen will burn to silicon which takes about 1 year. The silicon will burn to still heavier nuclei, ^{56}Fe, ^{56}Co, ^{56}Ni. But then the game is over. No more energy can be extracted via fusion. Now the entire central part of the star collapses which takes about 0.1 s. As the central part of the star has a mass that is larger than the Chandrasekhar mass, the collapse will continue until also the atomic nuclei form a degenerated Fermi gas. The collapse stops but the outer parts of the centrum will bounce and the collapse will turn into an explosion. In the central part of the star we still have the reaction

$$e^- + p \rightarrow n + \nu_e$$

This will be ireversible as the neutrinos will leave the star. An enormous number of neutrinos will be produced (10^{58}) and radiated during a time of about 10 s. The inner part of the star is transformed into a neutron star. The star dies in a *supernova*. The average energy of the neutrinos, as they escape from the star is of the order of 10 MeV. This means that the star emits an energy of some 10^{47} J which is as much energy as *the entire visible universe radiates during the same time*!

In the middle of an expanding gas cloud is the neutron star that can be observed as a *pulsar*. A pulsar can rotate 1000 turns a second! A neutron star with a mass equal to the mass of the sun will have a radius of about 10 km. In spite of the rapid rotation the neutron star is almost spherical.

Black holes

If the star mass is of the order of thirty solar masses or more, the shock wave cannot turn the implosion into an explosion. Then the degenerated nucleon gas will be relativistic and the star will collapse into a black hole with a radius that is uniquely determined by the mass

$$R = \frac{2GM}{c^2}$$

Review:
Give a rough sketch of the development of the universe: time, temperature, energy, density, particles. What are the problems with the simple Big Bang? What is inflation and how does it solve these problems? In what way is the ratio helium/ hydrogen a support for the Big Bang theory. Sketch how galaxies form, what determines their size? Sketch the life of a star: different results. What is a white dwarf, a neutron star, a pulsar, a black hole, a supernova, a brown dwarf, Chandrasekhar mass?

Problems:
1. Compute the size of a black hole with a mass equal to the mass of the earth.

2. Assume a typical pulsar with a mass equal to the mass of the sun and with a radius of 10 km, and a rotational period of 20 ms, i.e. 50 turns a second. A rotating body will have an equatorial radius r_e that is larger than the polar radius r_p because of the centrifugal force. This is called *oblateness* and is measured by the oblateness parameter $\varepsilon = \frac{r_e - r_p}{r_e}$. The earth, that rotates one turn a day, has ε = 1/300.
a) Estimate the oblateness of the pulsar. (0.2·1 / 300 !)
b) Estimate the rotational speed of a point on the equator, expressed in units of the speed of light.

3. Assume that the visible part of the universe only contained one galaxy, our own, consisting of about 10^{11} suns like our own (the sun is a rather good example of the average star). Assume that space-time is flat i. e. the density parameter is 1. What would then the age of our present universe be? (Of the order of 1 week!!)

Chapter 9. The dark matter

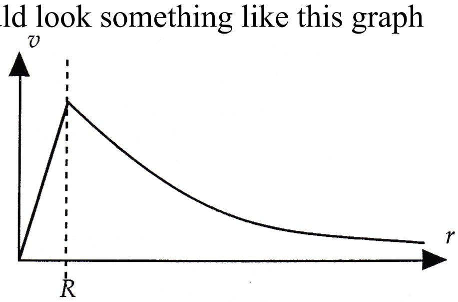
The visible galaxy consists of stars, dust and gas that rotate around a common centre under the influence of gravity. Most of the mass of the galaxy is concentrated in the inner parts, a situation very similar to our solar system. A star with mass *m* at distance *r* from the centre in a spherically symmetric galaxy and orbital speed *v* in a circular orbit then has

$$\frac{mv^2}{2} = G\frac{mM(r)}{r^2} \qquad \text{or} \qquad v = \sqrt{\frac{GM(r)}{r}}$$

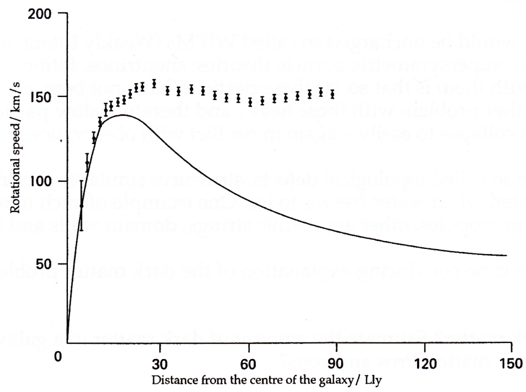
$M(r)$ is the mass inside the radius *r*. If we assume a constant mass density ρ inside a radius *R,* being zero outside this radius we have

$$v = \sqrt{\frac{4\pi G\rho}{3}}r \quad r < R \quad v = \sqrt{\frac{4\pi G\rho}{3}}R^{3/2}r^{-1/2} \quad r > R$$

The expected rotational speed of such a galaxy as a function of the distance from the centre would look something like this graph



The observed rotational curves look like this



The full curve is the expected orbital speed taking the visible mass into account. Instead of decreasing, the orbital speeds remain constant or even increase slightly out to very large distances from the galaxy centre. The conclusion is that the galaxy contains enormously much more matter that what can be seen, around the central concentration of visible matter there must be a spherical halo of dark, invisible matter that stretches very far out.

Assuming a constant speed v_c, as we go out to large distances, we see that this implies that

$$M(r) = \frac{rv_c^2}{G}$$

Further if the halo extends to a radius R_2 while the visible matter is contained within a radius R_1, we have that the visible mass is

$$M_1 = \frac{R_1 v_c^2}{G}$$

and the dark mass is roughly

$$M_2 = \frac{R_2 v_c^2}{G} \quad \text{or} \quad \frac{M_2}{M_1} = \frac{R_2}{R_1}$$

Observations indicate that the halo radius is about 30 times the galaxy radius, i.e. the dark mass has a mass of the order of 30 times the mass of the visible galaxy. This means that $\Omega_0 = 0.3$ instead of 0.01.

So what is then this dark matter? One candidate could be normal matter in brown dwarves and black holes. However, it turns out that the production of deuterium in the nuclear synthesis is very sensitive to the amount of protons and neutrons. Calculations done by David Schramm at the University of Chicago indicate that these candidates cannot contribute substantially to the density of the universe. The dark matter must be something else than normal matter.

Another candidate would be neutrinos. Recent observations (see appendix A) indicate that the neutrinos can have mass, especially so the, μ and τ neutrinos. The problem with this kind of matter is that as it moves very fast, the masses of the condensations that eventually collapse must be very large. This in turn means that galaxies in such a model will form late, in fact too late to fit our observations.

Still another candidate would be uncharged so called WIPMs (Weakly Interacting Massive Particles) that appear in supersymmetric particle theories: sneutrinos, fotinos, gluinos, zinos. The problem with them is that so far these particles have not been observed experimentally. Another problem with these heavy and therefore slow particles is that galaxy condensations collapse to easily — again in conflict with observations.

Other possibilities are so called topological defects, structures similar to the crystal domains that are created when water freezes to ice. One example of such topological defects are magnetic monopoles, other are cosmic strings, domain walls and textures. However, so far there is no convincing explanation of the dark matter problem.

Review:
Why do we need dark matter? Estimate the amount of dark matter in a galaxy.
Candidates for the dark matter: Pros and cons?

Problems:
1. In a typical galaxy the rotation speed increases with the distance from the centre of the galaxy but becomes constant for larger distances and of the order of magnitude some hundreds of km/ s.

Assume that the mean density of the entire galaxy is the same as the critical density. Estimate the size of such a typical galaxy. You can take the Hubble parameter to be 75 km/s/Mpc. (1 pc = 3.26 light-years).

2. The present Hubble parameter had (1998) an uncertainty factor that was 2. It is therefore often written in the form H = $100 \cdot h$ km /s /Mpc where h is a number between 0.5 and 1 and thus contain the uncertainty.

If you plot the rotation speed for matter in a typical galaxy as a function of the distance from the centre of the galaxy, you find that the speed increases toward a constant values that is of the order of 200 km/ s (see earlier figure). This can be interpreted such that the galaxy contains matter that we cannot see. Assume that this rotation speed is constant out to a radius R_{max} that is of the same order of magnitude as half the mean distance between the galaxies. The mean distance between the galaxies can be estimated to be $5 \cdot h^{-1}$ Mpc.

a) Show that the number of galaxies per volume is $0.01 \cdot h^3$ Mpc^{-3}.
b) Compute the density parameter Ω for the system above. (0.67)

Table of masses and decays of some hadrons

Hadron	*Mass / MeV / c^2*	*Lifetime / s*	*Common decay*
p	938.3	stable	
n	939.6	890	$pe^-\bar{\nu}_e$
Λ^0	1116	$2.6\cdot10^{-10}$	$p\pi^-$
Σ^+,Σ^-	1190	$\approx10^{-10}$	$p\pi, n\pi$
Σ^0	1190	$6\cdot10^{-20}$	$\Lambda^0\gamma$
Ξ^0,Ξ^-	1320	$\approx10^{-10}$	$\Lambda^0\pi$
Δ	1230	$5\cdot10^{-24}$	$p\pi, n\pi$
π^+,π^-	140	$2.6\cdot10^{-8}$	$\mu^+\nu_\mu, \mu^-\bar{\nu}_\mu$
π^0	135	$8\cdot10^{-17}$	$\gamma\gamma$
K^+,K^-	494	$1.2\cdot10^{-8}$	$\mu^+\nu_\mu, \mu^-\bar{\nu}_\mu$
$K^0,\bar{K}^0$	498	$10^{-10};10^{-8}$	$\pi\pi, \pi\pi\pi$

Table of masses and decays of some leptons

Lepton	*Mass / MeV / c^2*	*Lifetime / s*	*Common decay*
e^-	0.511	stable	
ν_e	0	stable	
ν_μ	0	stable	
μ^-	106	$2.2\cdot10^{-6}$	$e^-\bar{\nu}_e\nu_\mu$

Table of force particles

Particle	Mass/MeV/c^2	Spin	Force
γ	0	1	electromagnetic
W^+,W^-,Z^0	80000,91000	1	weak
8 *gluons*	0	1	strong
graviton	0	2	gravitational

Appendix A. Neutrino masses and neutrino oscillations

Assume that there is a mixing between the mu and tau neutrino states. This is very similar to the W^0-B mixing in the Standard Model. We write

$$\begin{aligned} \nu_\mu &= \nu_1 \cos\theta - \nu_2 \sin\theta \\ \nu_\tau &= \nu_1 \sin\theta - \nu_2 \cos\theta \end{aligned} \qquad \text{with some mixing angle } \theta .$$

We write this in matrix form that will somewhat simplify the calculations

$$\begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \text{ with inverse } \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Now let the neutrinos 1 and 2 propagate with a given momentum but assume they have different masses. The free plane wave solutions of the Schrodinger equation are

$$\nu_i(t) = \nu_i(0) e^{\frac{i}{\hbar}(px-Et)} \qquad i = 1,2$$

or in matrix form

$$\begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix} = e^{\frac{i}{\hbar}px} \begin{pmatrix} e^{-\frac{i}{\hbar}E_1 t} & 0 \\ 0 & e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix} \begin{pmatrix} \nu_1(0) \\ \nu_2(0) \end{pmatrix}$$

This gives immediately

$$\begin{pmatrix} \nu_\mu(t) \\ \nu_\tau(t) \end{pmatrix} = e^{\frac{i}{\hbar}px} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{-\frac{i}{\hbar}E_1 t} & 0 \\ 0 & e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_\mu(0) \\ \nu_\tau(0) \end{pmatrix}$$

or

$$\nu_\tau(t) = \nu_\mu(0)\cos\theta\sin\theta\left(e^{-\frac{i}{\hbar}E_1 t} - e^{-\frac{i}{\hbar}E_2 t} \right) + \nu_\tau(0)\left(\sin^2\theta e^{-\frac{i}{\hbar}E_1 t} + \cos^2\theta e^{-\frac{i}{\hbar}E_2 t} \right)$$

skipping the space part of the wave. Taking initial conditions

$$\nu_\mu(0) = 1 \quad \nu_\tau(0) = 0$$

we have

$$\nu_\tau(t) = \cos\theta\sin\theta\left(e^{-\frac{i}{\hbar}E_1 t} - e^{-\frac{i}{\hbar}E_2 t} \right) = \cos\theta\sin\theta e^{-\frac{i}{2\hbar}(E_1+E_2)t}\left(e^{\frac{i}{2\hbar}(E_2-E_1)t} - e^{-\frac{i}{2\hbar}(E_2-E_1)t} \right)$$

or

$$\nu_\tau(t) = 2i\cos\theta\sin\theta e^{-\frac{i}{\hbar}(E_1+E_2)t} \sin\frac{(E_2-E_1)t}{2\hbar}$$

Taking the modulus square give us a probavility

$$|\nu_\tau(t)|^2 = \sin^2 2\theta \sin^2 \frac{(E_2-E_1)t}{\hbar}$$

Now assuming $E_1, E_2 >> m_1, m_2$ we have

$$E_2 - E_1 = \sqrt{p^2c^2 + m_2^2c^4} - \sqrt{p^2c^2 + m_1^2c^4} = pc\left(1 + \frac{m_2^2c^4}{2p^2c^2} - 1 - \frac{m_1^2c^4}{2p^2c^2} \right) = \frac{\Delta(m^2)c^4}{2pc}$$

Using that $E \approx pc$ we arrive at $|\nu_\tau(t)|^2 = \sin^2 2\theta \sin^2 \frac{\Delta(m^2)c^4 t}{4E\hbar}$

Finally if the neutrino travels the distance L with about the speed of light c we put $t = L / c$ and have our final expression for the probability that a mu neutrino will have changed into a tau neutrino after having travelled the distance L:

$$P\left(\nu_\mu \to \nu_\tau\right) = \sin^2 2\theta \sin^2 \frac{\Delta\left(m^2\right)c^3 L}{4E\hbar}$$

If there is a mass difference between the electron neutrino and muon / tau neutrino we would also have oscillations between these neutrinos.

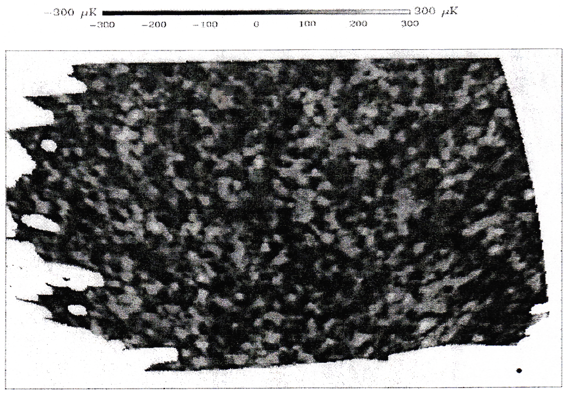
One indication of such neutrino oscillations is that when we measure the neutrino flux from the sun on the earth we detect only about 1/3 of the expected number. Also it has long been known that the ratio of mu / electron neutrino is different if they come from above (when they have travelled a short distance) than if they come from below (when they have travelled also through the earth). As the muons are produced from decays of pions we have the following decay chain

$$\pi^- \to \mu^- + \bar{\nu}_\mu \to e^- + \nu_\mu + \bar{\nu}_\mu + \bar{\nu}_e$$

Thus we would expect a ratio between mu neutrinos and electron neutrinos to be 2. In 1999 the Super-Kamiokande Collaboration reported measurements that definitely showed neutrino oscillations.

These measurements give a value of $\sin^2 2\theta \approx 1$ and $\Delta\left(m^2\right) \approx 10^{-3}$ eV for $\nu_\mu \leftrightarrow \nu_\tau$. Latest observations also support conversions $\nu_\tau \leftrightarrow \nu_e$ and with small probability also $\nu_\mu \leftrightarrow \nu_e$. The mass difference between the muon and electron neutrinos seems to be very small.

Appendix B. Fluctuations in the microwave background radiation.



Recent more accurate measurements of the temperature fluctuations in the cosmic microwave radiation (se picture) give support for the Big Bang model and also gives a clear indication the density parameter Ω is 1, that the universe is flat.

Theory:
The speed of sound in a photon gas is given by

$$v = \sqrt{p/\rho}$$

where p is the pressure and ρ the density. Further it can be shown that the pressure in a photon gas is given by $p = \frac{1}{3}\rho c^3$ giving

$$v = c/\sqrt{3}$$

If we imagine standing sound waves in the photon-baryon medium with a wavelength of λ and a period P we then have

$$\lambda \cdot f = \lambda / P = c/\sqrt{3}$$

The distance between compressions in the standing wave is $d = \lambda / 2$. In order for the longest waves to have made a compression at the decoupling time t_d we must have $t_d = P/4$. Using these facts we arrive at

$$d = 2ct_d/\sqrt{3}$$

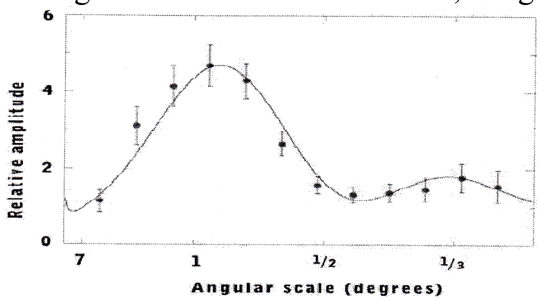
for the largest distance between compressions (which will have a slightly higher temperature). This distance will have followed the general expansion in the following matter dominated universe and the present size will then be

$$d_0 = d\left(\frac{t_0}{t_d}\right)^{2/3} = 2ct_d\left(\frac{t_0}{t_d}\right)^{2/3}/\sqrt{3}$$

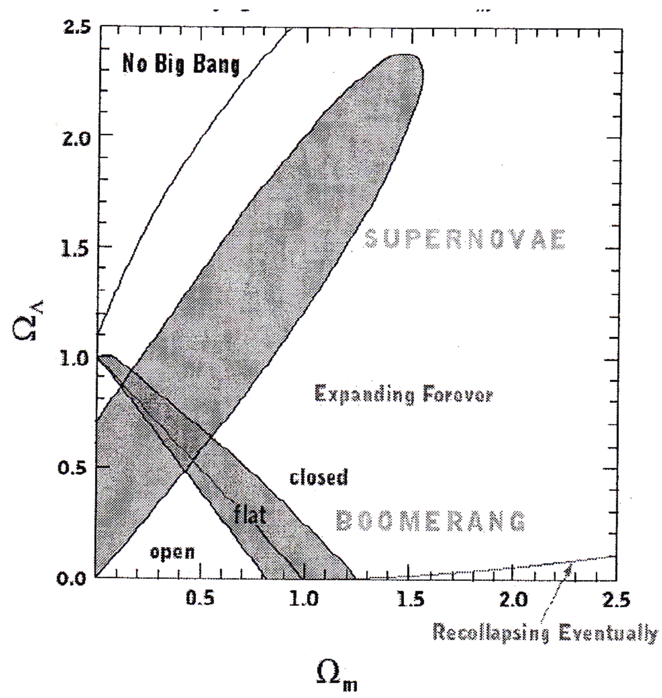
The distance between an observer and the visible horizon is given $D_0 = 3ct_0$ which means that the angular size of the temperature fluctuations will be

$$\frac{d_0}{D_0} = d\left(\frac{t_0}{t_d}\right)^{2/3} = \frac{2}{3\sqrt{3}}\left(\frac{t_0}{t_d}\right)^{1/3}(radians) = \frac{120}{\pi\sqrt{3}}\left(\frac{t_0}{t_d}\right)^{1/3}(degrees)$$

Inserting numbers $t_0 = 13 \cdot 10^9$ years and $t_d = 3 \cdot 10^5$ years, about 1°. This is the same order of magnitude as the observed value, see graph below.



Combining this result with the data from distant supernovas we get a prediction for the matter part of the density parameter of $\Omega_m = 0.3$ and for the vacuum density $\Omega_\Lambda = 0.7$.



Appendix C. The distance ladder

One of the big problems in observational cosmology is to measure the distance to objects that are far away. The velocity can be measured more easily via the red-shift.

The usual approach, which is the same as that developed by Hubble, is to construct a so called *distance ladder*. Relative distance measures are used to establish each "rung" of the ladder and calibrating these measures against each other allows one to measure distances up to the top of the ladder. A modern analysis might use several rungs, based on different distance measures.

First, one exploits local kinematic distance measures to establish the length scale within the galaxy. Kinematic methods do not rely upon knowledge of the absolute luminosity of a source. Distances up to 30 pc can be derived using the trigonometric parallax of a star, i.e. the change in angular position of a star on the sky in the course of a year due to the earth's motion in space. There are several other parallaxes in addition, such that one can measure distances in this way up to a few hundred parsecs.

Once one has determined distances of nearby stars with the kinematic method, one can then calculate their absolute luminosity (L) from their apparent luminosity (l) and their known distance (r): $L = 4\pi r^2 l$. In this way it was learnt that most stars have a strict relationship between spectral type (a indicator of surface temperature) and absolute luminosity. With this method one can measure the distance of stars of known apparent luminosity and spectral type and in this way measure distances up to around 30 kpc.

Another important class of distance indicators contain variable stars of various kinds, including *Classical Cepheids*. These are bright variable stars which have a very strict relationship between the period of variation P and their absolute luminosity: $\log P \propto \log L$.

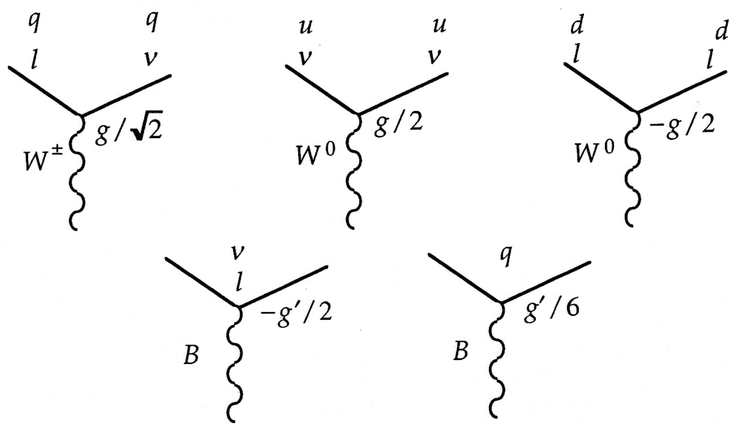
The measurement of the period of a distant Cepheid thus allows one to determine its distance. These stars are so bright that they can be seen in galaxies outside our own and they extend the distance scale up to around 4 Mpc. Other distance indicators based on *novae*, *blue super-giants* and *red super-giants* allow the ladder to be extended slightly to around 10 Mpc. These are called *primary* distance indicators.

The *tertiary distance indicators* include the brightest cluster galaxies and supernovae. With the brightest galaxies one can reach distances of several hundred Mpc. Supernovae are stars that explode, producing a luminosity roughly equal to that of an entire galaxy. These stars are therefore easily seen in distant galaxies, but these indicators are not too precise.

There are many more other methods to estimate the distance of galaxies that could be described here. If possible several methods are combined to give the smallest experimental error. There are new satellite projects planned, like the GAIA misson, that is proposed by ESA (*European Space Agency*) for 2010. GAIA will measure more than 10^9 stars in our galaxy and its nearest neighbours. With such future measurements almost the whole galaxy will have trigonometrically determined distances.

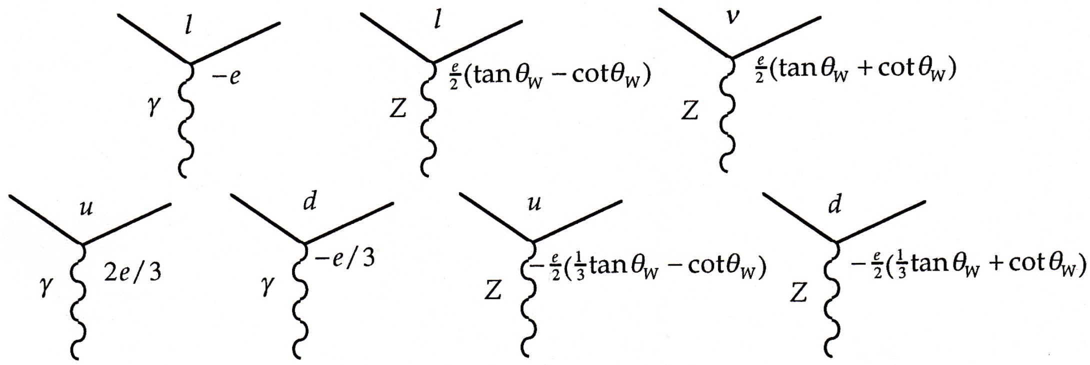
You can find more about the GAIA project on http://astro.estec.esa.nl/SA-general/Projects/GAIA/gaia.html

Fundamental couplings (left-handed)



$$\gamma = B\cos\theta_W + W^0\sin\theta_W$$
$$Z = -B\sin\theta_W + W^0\cos\theta_W$$
$$g = \frac{e}{\sin\theta_W} \qquad g' = \frac{e}{\cos\theta_W}$$

Derived couplings



Right-handed particles/Left-handed anti-particles